2

Mathematical Tools & Vector

CONTENTS

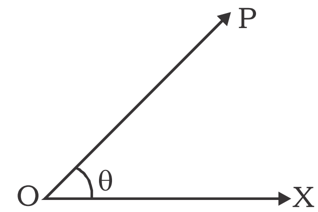
- *Trigonometry*
- *Quadratic Equation*
- *Logarithms*
- *Co-ordinate Geometry*
- *Progressions*
- *Graphs*
- *Calculus*
- *Vectors*

AIPMT SYLLABUS

Elementary concepts of differentiation and integration for describing motion. *Scalar and vector quantities:* Position and displacement vectors and notation, equality of vectors, multiplication of vectors by a real number; addition and subtraction of vectors.

INTRODUCTION

ANGLE



Consider a revolving line OP. Suppose that it revoloves in anticlockwise direction stating from its initial position OX,

The angle is defined as the amount of revolution that the revolving line makes with with its initial position.

From fig. the angle covered by the revolving line OP is given by $\theta = \angle POX$

The angle is positive, if it is traced by the revolving line in anticlockwise direction and is negative, if it is covered in clockwise direction

1 right angle = 90° (degrees)

1° = 60° (minutes)

1′ = 60° (second)

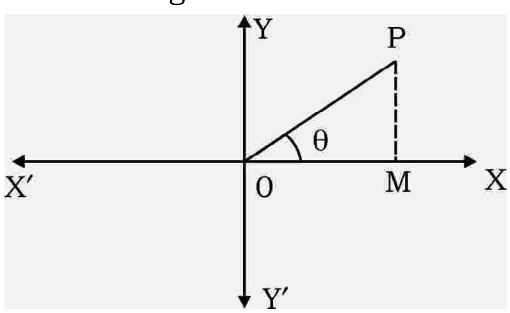
In circular system I right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle substended at the centre of a cirlce by an arc of the circle, whose length is equal to the radius of the circle.

1 rad = 180° / π ≈ 57° 17 45 ≈ 54.3°

TRIGONOMETRIAL RATIOS (OR T RATIOS) :

Consider the two fixed line XOX′ and YOY′ intersecting at right angles to each other at point O as shown in figure then.



1. Point O is called origin.
2. XOX′ and YOY′ are known are X-axis and Y-axis respectively
3. Portions XOY, YOX′, X′OY′ and Y′OX are called I, II, III and IV quadrant respectively.

Consider that the revolving line OP has traced out angle θ (in I quadrant) in anticlockwise direction.

From P, drop PM perpendicular to OX. Then, side OP (in front of right angle) is called hypotenuse, side MP (in front of angle θ) is called opposite side or perpendicular and side OM (making angle θ with hypotenuse) is called adjacent side or base.

The three sides of a right angle triangle are connected to reach other through six different ratios, called trigonometric ratios or simply T - ratios. These are defined as bleow :

1. Sin θ = $\frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP}$ (read as sine of angle θ)
2. Cos θ = $\frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP}$ (read as cosine of angle θ)
3. Tan θ = $\frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM}$ (read an tangent of angle θ)
4. Cot θ = $\frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}$ (read as cotangent of angle θ)
5. Sec θ = $\frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM}$ (read as secant of angle θ)
6. Cosec θ = $\frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}$ (read as consecant of angle θ)

It can be easily proved that :

1. (a) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ (b) $\sec\theta = \frac{1}{\cos\theta}$ (c) $\cot\theta = \frac{1}{\tan\theta}$
2. (a) $\sin^2\theta + \cos^2\theta = 1$ (b) $1 + \tan^2\theta = \sec^2\theta$ (c) $1 + \cot^2\theta = \text{cosec}^2\theta$

TRIGONOMETRIC IDENTITIES :

$\sin^2\theta + \cos^2\theta = 1$; $\tan^2\theta = \sec^2\theta - 1$;

$\cot^2\theta = \text{cosec}^2\theta - 1$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1$

$\cos^2 A = \frac{1 + \cos 2A}{2}$

$\sin^2 A = \frac{1 - \cos 2A}{2}$

$2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$2\sin A \sin B = \cos(A - B) - \cos(A + B)$

If $\sin\theta = x$, then $\theta = \sin^{-1}(x)$

If $\tan\theta = x$, then $\theta = \tan^{-1}(x)$

$-1 \le \sin\theta \le 1$ always

$-1 \le \cos\theta \le 1$ always

$-\infty \le \tan\theta \le \infty$ always

$-1 \ge \operatorname{cosec}\theta \ge 1$ always

$-1 \ge \sec\theta \ge 1$ always

$-\infty \le \cot\theta \le \infty$ always

QUADRATIC EQUATION

$ax^2 + bx + c = 0$ is a quadratic equation in x

$\underline{x} = -\frac{b \pm \sqrt{D}}{2a}$, where $D = \sqrt{b^2 - 4ac}$

(a,b)

where a, b are two roots of equation

$a + b = -\frac{b}{a}$; $ab = \frac{c}{a}$

if D < 0, no real roots

if D = 0, one real root (equal roots)

if D > 0, 2 real roots

LOGARITHMS

$\log_e x \rightarrow \ln x$

$\log_a x + \log_a y = \log_a xy$

$\log_a x - \log_a y = \log_a \frac{x}{y}$

$\log_a x^m = m \log_a x$

$\log_a b = \frac{\log_c b}{\log_c a}$

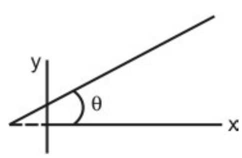
where c is any +ve no.

$\log_a b = \log_b a$

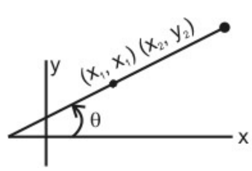
CO-ORDINATE GEOMETRY

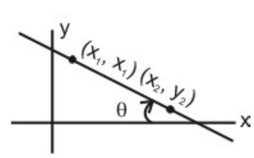
Slope of a line (m)

= tan of $\angle$ made with +ve × axis = $\tan\theta$



Slope $(m) = \tan\theta$



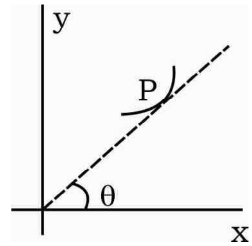


Slope $m = -\tan\theta$

$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

St. Line has constant slope

Slope of a curve means slope of tangent at that point (slope at P = $\tan\theta$)



Area of trapezium

$= \frac{1}{2}$(sum of parallel side) x altitude

PROGRESSIONS

a, a + d, a + 2d, a + 3d .... is an AP

n^{th} term of an AP = a + (n – 1) × d

Where a is first term d is common difference

Sum of n terms of AP

S_n = a + 2 (n – 1) × d

a, ar, ar^2, ar^3, ... is a GP

$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

$a + ar + ar^2 + ar^3 + ... \infty = \frac{a}{1-r}$ (if r <1)

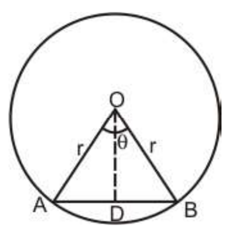
Some Basic informations

OD is perpendicular drawn on AB from point O.

$\frac{AD}{AO} = \sin\frac{q}{2}$

$AD = r\sin\frac{q}{2}$

Chord AB = 2r sin $\frac{q}{2}$



Sum of interior ∠S of a polygon of n sides = (n–2) 180°

Sum of exterior ∠S of a polygon of (n sides) = 360°

Exterior ∠ of a regular polygon of n sides = $\frac{360°}{n}$

CHECK YOUR GRASP

☞ Find the value of

(i)	sin 30° + cos 60°	(ii)	sin 0° – cos 0°	(iii)	tan 45° – tan 37°
(iv)	sin 150°	(v)	cos 120°	(vi)	tan 135°
(vii)	cos 300°	(viii)	sin (– 30°)		
(ix)	cos (– 60°)	(x)	tan (– 45°)		

☞ Find the value of

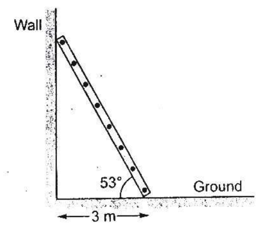
(i) 3 + 6 + 12 + ... + 60

(ii) 2 + 4 + 8 + 16 + 32 + ...+ 512

(iii) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....\infty$

EVALUATE YOUR SELF

1. Bottom end of a ladder leaning against a wall is 3 m away from the foot of the wall as shown in figure, length of the ladder is:



(1) 8 m (2) 5 m (3) 4 m (4) 6 m

2. Select incorrect alternative

(1) $\sin 37° = \frac{3}{5}$ (2) $\sin 53° = \frac{4}{5}$ (3) $\tan 37° = \frac{4}{3}$ (4) $\cos 30° = \frac{\sqrt{3}}{2}$

3. As x increases from θ to $\frac{\pi}{2}$, the value of sin x:

(1) increases (2) decreases

(3) remains constant (4) first increases then decreases

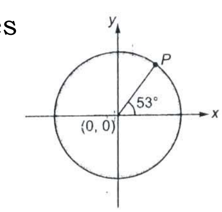
4. The value of $(\sin 180° + \cos 90°)^2$ is :

(1) 4 (2) 2 (3) 1 (4) 0

5. The length of a string between a kite and a point on the ground is 55 m. If the string makes an angle of 53° from level ground and there is no slack in the string, then the height of the kite is:

(1) 55 m (2) 44 m (3) 33 m (4) 11 m

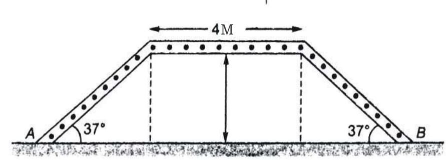
6. A circle of radius 10 units is shown in figure. The coordinates of point P is:



(1) (6, 8) (2) (8, 6)

(3) (10, 10) (4) can't be determined

7. A schematic diagram of a pedestraian overpass is shown in figure. If you walk on the overpass from point a to point B, how far have you walked? The Maximum height of the overpass is 12 m.



(1) 60 m (2) 44 m (3) 64 m (4) 100 m

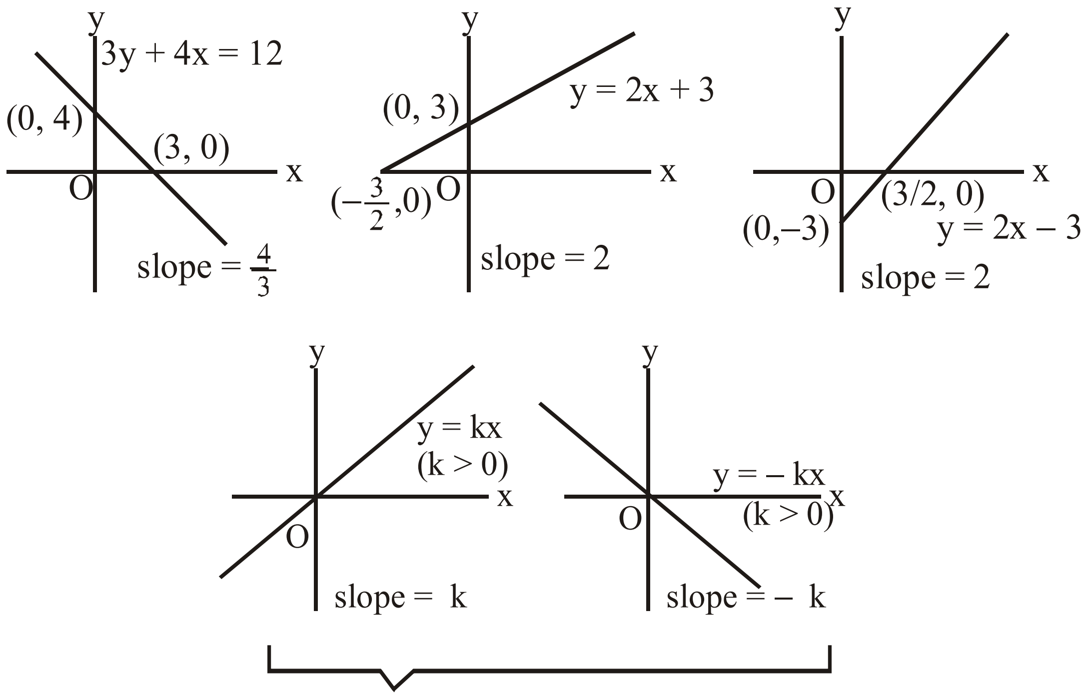
ANSWER

1. (2) 2. (3) 3. (1) 4. (4) 5. (2)

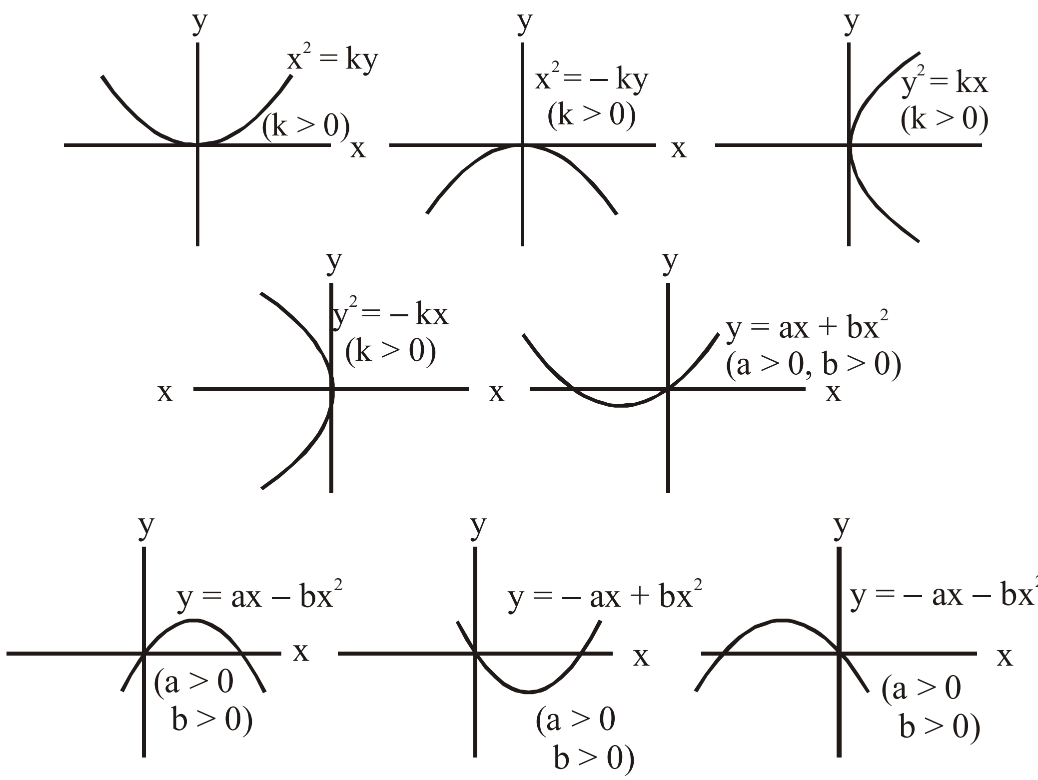
6. (1) 7. (2)

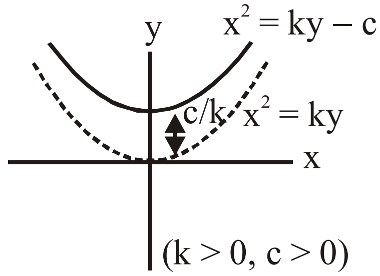
GRAPHS

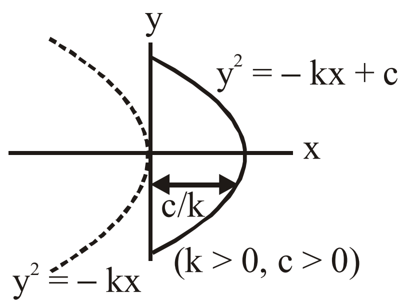
1. Straight lines [linear equations]



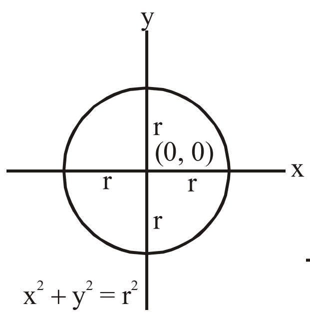
2. Parabolas



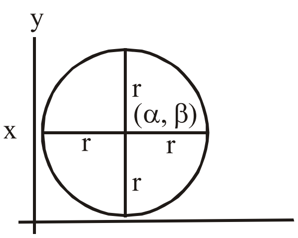




3. Circles

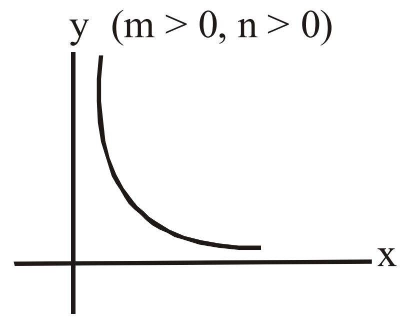


$x^2 + y^2 = r^2$
(standard circle)

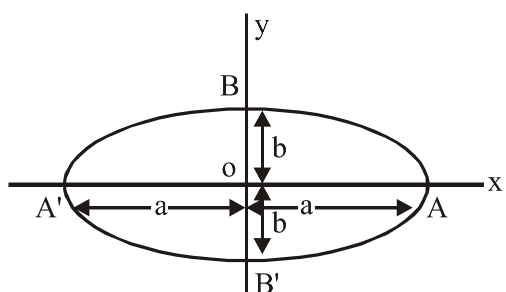


$(x - \alpha)^2 + (y - \beta)^2 = r^2$
general circle

4. Inverse relation $y^m \propto \frac{1}{x^n}$

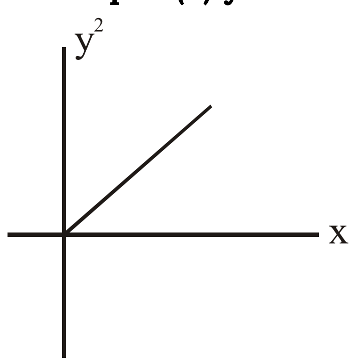


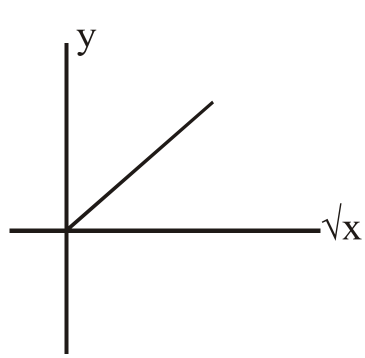
5. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



B'B → minor axis
OB → semi minor axis
OA → semi major axis
A'A → major axis

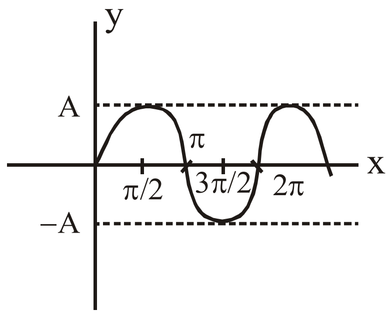
6. Other Graphs (a) $y^2 = 4x$



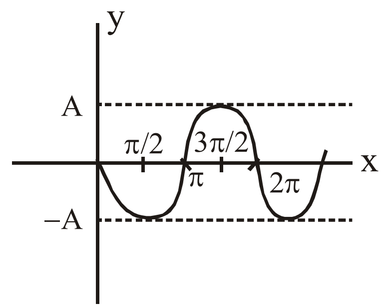


7. Trignometric:

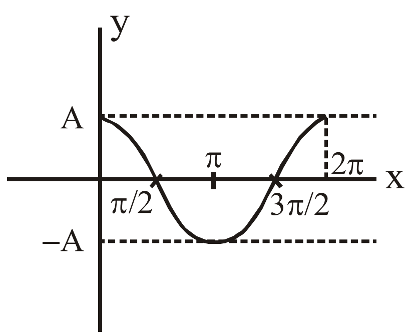
(a) $y = A \sin x$



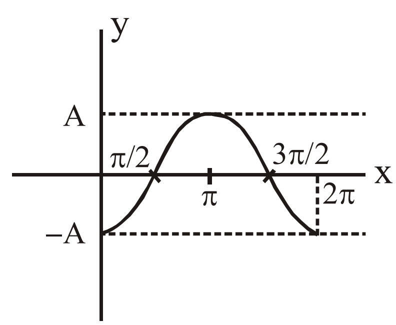
(b) $y = -A \sin x$



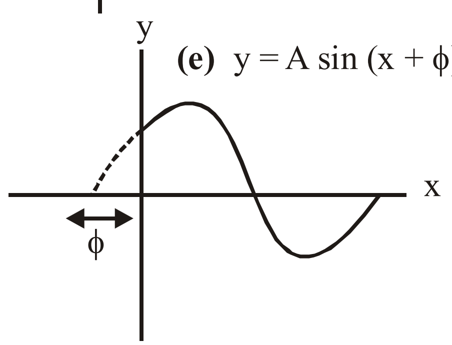
(c) $y = A \cos x$



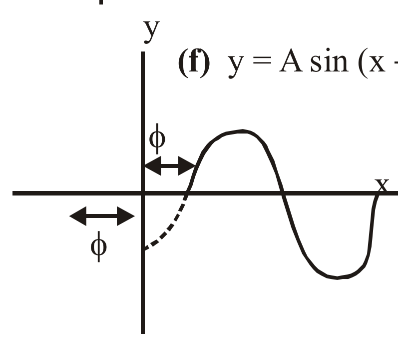
(d) $y = -A \cos x$



(e) $y = A \sin (x + \phi)$



(f) $y = A \sin (x - \phi)$



CHECK YOUR GRASP

☞ *What is the locus of path represented by $y = 2x + 3$?*

☞ Find the slope of line represented by equation $2x - 3y = 4$?

EVALUATE YOURSELF - 2

Draw the following graphs [a v/s b means a → y –axis, b → x –axis]

1. v v/s t for $v = u + at$, $v = u - at$, $v = at$

   s v/s t for $v = -at$, $v = -at$

2. s v/s t for $s = \pm ut \pm \frac{1}{2}at^2$, $s = \pm \frac{1}{2}at^2$

3. v v/s s for $v^2 = u^2 \pm 2as$

4. $\left.\begin{array}{l} k \text{ v/s } v \\ k \text{ v/s } v^2 \\ \sqrt{k} \text{ v/s } v \end{array}\right\}$ for k = (1/2) mv2

5. y v/s t for y = A sin (wt ± f)

6. y v/s x for y = x tan θ – 1/2g x2/u2 sec2q

7. F v/s r for $F = \frac{G m_1 m_2}{r^2}$

F v/s 1/r

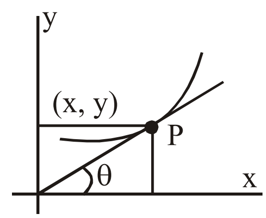
CALCULUS

DIFFERENTIAL CALCULUS ($\frac{dy}{dx}$ AS RATE MEASURE)

The derivative or differential coefficient of a function is the limit to which the ratio of the small increment in the function to the corresponding small increment in the variable (on which it depends) tends to, when the small increment in the variable approaches zero.

$\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$ = Instantaneous change of y with respect to x

It is also equal to the slope of the tangent drawn at the point where dy/dx is to be calculated



$\frac{dy}{dx} = \tan\theta$

Formula for calculating $\frac{dy}{dx}$ due to different functions

(1) $y = x^n$; $\frac{dy}{dx} = nx^{n-1}$

(2) y = sin x ; $\frac{dy}{dx} = \cos x$

(3) y = cos x, $\frac{dy}{dx} = -\sin x$

(4) y = tan x ; $\frac{dy}{dx} = \sec^2 x$

(5) $y = \cot x ; \frac{dy}{dx} = -\text{cosec}^2 x$

(6) $y = \sec x ; \frac{dy}{dx} = \sec x \tan x$

(7) $y = \text{cosec } x ; \frac{dy}{dx} = -\text{cosec } x \cot x$

(8) $y = e^x$; $\frac{dy}{dx} = e^x$

(9) $y = \log_e x$; $\frac{dy}{dx} = \frac{1}{x}$

(10) $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = -\frac{1}{f^2(x)}.\frac{d}{dx}(f(x))$

(11) $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

RULES

(1) If y = k f (x), where k is a constant

Then, $\frac{dy}{dx} = k\frac{d}{dx}[f(x)]$

(2) If y = constant

Then, $\frac{dy}{dx} = 0$

(3) If $y = u \pm v \pm w$, where u, v and w are functions of x

Then, $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$

(4) If y = u.v

Then, $\frac{dy}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

(5) If y = u/v

Then, $\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$

CHECK YOUR GRASP

- ☞ *Find dy/dx, if* $y = x^{3/2}$
- ☞ *Find dy/dx, if* $y = \sin 2x$
- ☞ *Find dy/dx, if* $y = e^{\sin x}$

EXAMPLES EXPLAINING DIFFERENT METHODS OF DIFFERENTIATION

(1) FUNCTION OF FUNCTION – CHAIN RULE

If $y = f(x)$ and x is a function of some other variable z. Then $\frac{dy}{dx}$ can be written as the product of two derivatives.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

Example 1 : $\frac{dy}{dx}\cos(\log x^n)$

Solution : $\frac{dy}{dx} = \frac{-\sin[\log x^n]}{x^n} \times n\, x^{n-1}$

Example 2 : $\sqrt{\sin(\log x)}\ \frac{dy}{dx}$

Solution : $\frac{dy}{dx} = \frac{1\cos(\log x)}{2\sqrt{\sin\log x}} \times \frac{1}{x}$

(2) Parametric Form

$x = f(t)\ ;\ y = f(t)$

where, t is a parameter

Then, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Example 3 : $\left.\begin{matrix} y = a\sin t \\ x = a\cos t \end{matrix}\right\} \frac{dy}{dx}$

Solution :

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dx}{dt} = -a\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\cot(t)$$

(3) Double Differentiation or Second Derivative

The second derivative of y whith respect to x is defined as the derivative of the function $\frac{dy}{dx}$ (or the derivative of the derivative). It is usually written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 4 : $x = t^3 + t^2 + t$. Find $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$

Solution : $\frac{dx}{dt} = 3t^2 + 2t + 1 \qquad \frac{d^2x}{dt^2} = 6t + 2$

Applications of Differentiation in Physics (To be Studied later)

(i) Instantaneous velocity $V_{ins} = \frac{dx}{dt}$

(ii) Instantaneous acceleration $a_{ins} = \frac{dv}{dt}$

also $a_{ins} = v\frac{dv}{dx}$

(iii) Instantaneous force $= F_{ins} = \frac{dp}{dt}$

also $F_{ins} = v\frac{dp}{dx} = \frac{p}{m}\frac{dv}{dx}$

(iv) Instantaneous power $= P_{ins} = \frac{dw}{dt}$

also $P_{ins} = v\frac{dw}{dx}$

(v) Instantaneous current $= i_{ins} = \frac{dq}{dt}$

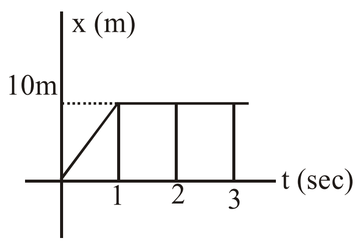
(vi) $F_v = -\eta A\frac{dv}{dx}$; (where, F_v = viscous force)

(vii) $F = \frac{-dU}{dx}$; (where, F = conservative force)

$\frac{dy}{dt}$ = rate of change of y w.r.t x, $\frac{dy}{dx}$ = gradient y.

Example 5 :

Find the velocity of a particle whose displacement time curve is shown at t = 0.5s and t = 2.5s.



Solution :

Vel at 0.5 sec = 10 m/s
Vel at t = 2.5 sec = 0 m/s } (slope)

Example 6 :

The position of a particle travelling along x-axis is given by equation $x = \frac{t^3}{3} - \frac{3t^2}{2} + 2t\ (m)$. Find

(a) Initial velocity of particle.

(b) Find acceleration of the particle when it at rest.

(c) Find the velocity of the particle when it is at equilibrium

Solution :

$$\frac{dx}{dt} = t^2 - 3t + 2 \quad \Rightarrow \quad \text{at } t = 0, v = 2$$

(b) Rest, means v = 0

$t^2 - 3t + 2 = 0$

$t = 1s$ $\quad$ $t = 2s$

Now,

$a = dv/dt = 2t - 3$

at $t = 1s$, $a = -1\ m/s^2$ $\quad$ at $t = 2s$, $a = 1 m/s$

(c) $F = 0 \quad \Rightarrow \quad a = 0$

$2t - 3 = 0 \; t = \frac{3}{2}$ sec

$$v_{t=\frac{3}{2}} = \left(\frac{3}{2}\right)^2 - \frac{3\times 3}{2} + 2 = \frac{9}{4} - \frac{9}{2} + 2 = \frac{-1}{4}\ m/s$$

Example 7 :

The P.E. of a particle as a function of its position is given by $U = ax^2 - b x$. Find the minimum P.E. (Minimum P.E. is at equilibrium)

Solution :

$$F = \frac{-dU}{dx} \Rightarrow -2ax + b = 0 \quad \Rightarrow x = \frac{-b}{2a}$$

At this position, PE is given by

$$U = a \times \left(\frac{-b}{2a}\right)^2 - b\left(\frac{b}{2a}\right) = a \times \frac{-b^2}{4a} - \frac{b^2}{2a}$$

$$U_{min} = \frac{b^2}{4a} - \frac{b^2}{2a} = \frac{-b^2}{4a}$$

Example 8:

The radius of a circle is increasing at a rate of 1 cm/sec. Find the rate of increase of its area, when its radius is 10 cm.

Solution :

$\frac{dA}{dt} = ?$ $\qquad$ $\frac{dr}{dt} = 1 cm/s$

(at r = 10 cm)

Now, $A = \pi r^2$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt} \text{ (at } r = 10)$$

$= \pi \times 2 \times 10 \times 1 = \pi \times 20 = 20\pi\ cm^2/s$

CHECK YOUR GRASP

☞ *Which physical quantity is obtained by the differentiation of position vector w.r.t time.?*

☞ *Which physical quantity is obtained by the differentiation of velocity taking time as the variable?*

☞ *Which physical quantity is obtained by the integration of velocity with time as variable?*

EVALUATE YOURSELF -3

Evaluate $\frac{dy}{dx}$

1. $y=x^2$
2. $y=\sqrt{x}$
3. $y = x$
4. $y = 3\sin x$
5. $y = e^x - \cos x + \log x$
6. $y = e^x - \tan x + \log_e x$
7. $y = e^x \cos x$
8. $y = \frac{\log x}{\tan x}$
9. $ax^4 + bx^3 + cx^2 + dx + e$
10. $(x-1)(x^2+2)$

11. The displacement of a particle is given by $y = a + bt + ct^2 - dt^4$. The initial velocity and acceleration are respectively

(1) b , $-4d$ (2) $-b$, $2c$ (3) b, $2c$ (4) $2c$, $-4d$

12. If y is function of x then $\frac{dy}{dx}$ measures

(1) Ratio of the change in y to the change in x between two points

(2) Ratio of y to x

(3) Change in y with respect to change in x at a particular point

(4) Change in y per unit change in x

13. If $y = 5x^4 + \cos 2x - e^{3x}$ the $\frac{dy}{dx}$ is

(1) $\frac{5}{4}x^3 - \frac{\sin 2x}{2} - \frac{e^{3x}}{3}$ (2) $0 + \sin 2x - e^{3x}$

(3) $20x^3 + 2\sin 2x - 3e^{3x}$ (4) $20x^3 - 2\sin 2x - 3e^{3x}$

14. The displacement of a particle along y axis is given by y = a coswt where a and w are constants then its acceleration along y axis will be

(1) $-aw^2 \sin wt$ (2) $-aw^2 \cos wt$ (3) $aw^2 \sin wt$ (4) $aw^2 \cos wt$

ANSWERS

1. $\frac{d}{dx}(x^2) = 2x$

2. $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{(1/2)}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

3. $\frac{d}{dx}(x) = 1.x^{\circ} = 1$ 4. $\frac{dy}{dx} = 3\cos x$

5. $\frac{dy}{dx} = e^x + \sin x + \frac{1}{x}$ 6. $\frac{dy}{dx} = e^x - \sec^2 x + \frac{1}{x}$

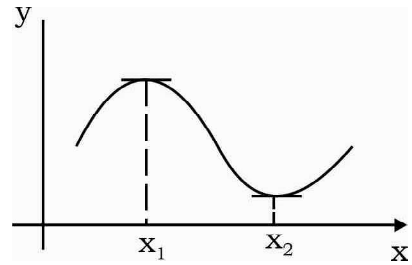
7. $\frac{dy}{dx} = -e^x \sin x + e^x \cos x$ 8. $\frac{dy}{dx} = \frac{(\tan x)\frac{1}{x} - \log x(\sec^2 x)}{\tan^2 x}$

9. $4ax^3 + 3bx^2 + 2cx + d$ 10. $3x^2 - 2x + 2$

11. (3) 12. (3) 13. (4) 14. (2)

MAXIMA AND MINIMA

Suppose a quantity y depends on another quantity x in a manner shown in figure. It becomes maximum at x_1 and minimum at x_2.



At these points the tangent to the curve is parallel to the X - axis and hence its slope is zero ie $\tan\theta = 0$. But the slope of curve $y - x$ equals the rate of change of y w.r.t. x is $\frac{dy}{dx}$.

Thus, at a maximum or minimum $\frac{dy}{dx} = 0$.

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative, at a maximum i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at a maximum.

The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$.

Thus the condition of a maximum is, $\left.\frac{dy}{dx} = 0 \quad \frac{d^2y}{dx^2} < 0 \right) - \text{maximum}$.

Similarly, at a minimum the slope changes from negative to positive. The slope increases at such a point and hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$. The condition of a minimum is $\left.\frac{dy}{dx} = 0 \quad \frac{d^2y}{dx^2} > 0 \right) - \text{minimum}$

Quite often it is well understood from the physical situation whether the quantity is a maximum or a minimum.

The test on $\frac{d^2y}{dx^2}$ may then be omitted.

Example 9 :

If $y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 1$, find maximum and minimum value of y and x corresponding to y (maximum) and y (minimum)

Solution :

For y_{max}

or y_{min} $\frac{dy}{dx} = 0$ $\Rightarrow$ $x^2 - 3x + 2 = 0$

$x = 1$ $\quad x = 2$

Now, $\frac{d^2y}{dx^2} = 2x - 3$

$\frac{d^2y}{dx^2} = -ve$ and $\frac{d^2y}{dx^2} = +ve$

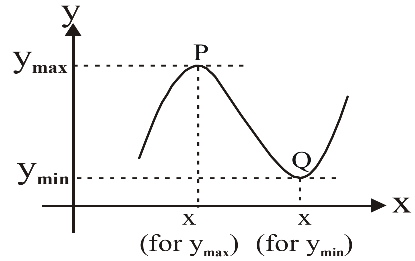
(at x = 1) (at x = 2)

$\therefore$ y is max at x = 1

and y is min at x = 2

$y_{max} = \frac{1}{3} - \frac{3 \times 1^2}{2} + 2 \times 1 + 1$

$y_{min} = \frac{2}{3} - \frac{3 \times 2^2}{2} + 2 \times 2 + 1$



at P ; $\frac{dy}{dx} = 0$; $\frac{d^2y}{dx^2} = -ve$

at Q ; $\frac{dy}{dx} = 0$; $\frac{d^2y}{dx^2} = +ve$

Example 10 :

Find y (maximum) and y (minimum) for the function $y = \frac{x^2}{2} - x$.

Solution :

$\frac{dy}{dx} = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

Now, $d^2y/dx^2 = 1 > 0$ $\therefore$ (only minima exists)

$y_{min} = \frac{1}{2} - 1 = \frac{1-2}{2} = -\frac{1}{2}$

CHECK YOUR GRASP

- *If y = 2x + 3, find the slope at x = 2.*
- *If $y = x^3 - 3x + 4$, find the slope at x = 1.*
- *Find the maximum value of y, if $y = 1 - x^2$.*
- *If the velocity of the particle is given by $v = t^2 - 2t + 4$ m/s. then*

(1) V_{max} = 3 m/s at t = 1 sec

(2) V_{min} = 3 m/s at t = 1 sec

(3) V_{max} = 1 m/s at t = 2 sec

(4) V_{max} = 4 m/s at t = 2 sec

INTEGRATION

If $y = F(x)$

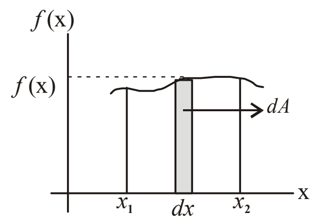
and $\frac{dy}{dx} = f(x)$

$y_2 - y_1 = \int_{y_1}^{y_2} dy = \int_{x_1}^{x_2} f(x)\,dx$ = total change in y as x changes from x_1 to x_2

$[F(x)]_{x_1}^{x_2} = F(x_2) - F(x_1)$

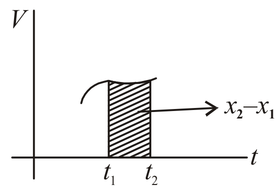
GEOMETRICAL MEANING OF INTEGRATION

Area under $f(x)$ vs. x-graph from x_1 to x_2 $A = \int_{x_1}^{x_2} f(x)\,dx$

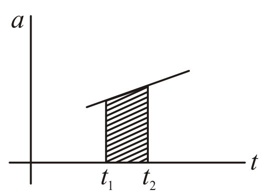


APPLICATION OF INTEGRATION PHYSICS

(a) $\because\ V = \frac{dx}{dt} \Rightarrow x_2 - x_1 = \int_{t_1}^{t_2} V dt$



(b) $a = \frac{dV}{dt} \Rightarrow V_2 - V_1 = \int_{t_1}^{t_2} a\,dt$

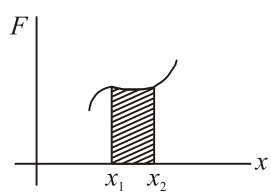


(c) $F = \frac{dP}{dt} \Rightarrow p_2 - p_1 = J = \int_{t_1}^{t_2} F\,dt$

(d) $W = \int_{t_1}^{t_2} P\,dt$

(e) $F = -\frac{dU}{dx} \Rightarrow -\int_{U_1}^{U_2} dU = \int_{x_1}^{x_2} F\,dx$

$W = U_1 - U_2 = \int_{x_1}^{x_2} F\,dx$



(f) $i = \frac{dq}{dt} \Rightarrow q_2 - q_1 = \int_{t_1}^{t_2} i\,dt$

FORMULAE IN INTEGRATION

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$\int \cos x\,.\,dx = \sin x + C$

$\int \sin x\,.\,dx = -\cos x + C$

$\int \sec^2 x\,.\,dx = \tan x + C$

$\int \text{cosec}^2 x.\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \text{cosec } x \cot x \, dx = -\text{cosec } x + C$ $\int e^x \, dx = e^x + C$

$\int \frac{1}{x} \, dx = \log_e x + C$

RULES FOR INTEGRATION

$\int (U \pm V \pm W) \, dx = \int U dx \pm \int V dx \pm \int W dx$

$\int nf(x) \, dx = n \int f(x) \, dx$ (where n = constant)

Example 12 :

$\int_1^2 (3x^2 + 2x + 1) \, dx$

Solution :

$\left.\frac{3x^{2+1}}{3} + \frac{2x^{1+1}}{2} + x\right|_1^2 = \left[x^3 + x^2 + x\right]_1^2 = (8 + 4 + 2) - (1 + 1 + 1) = 11$

Example 14 :

$\int_0^{\pi/4} \sin x \, dx$

Solution :

$\left[-\cos x\right]_0^{\pi/4} = -\frac{1}{\sqrt{2}}$

Example 14 :

$\int \sec(2x + 3) \tan(2x + 3) \, dx$

Solution :

$\frac{\sec(2x + 3)}{2} + C$

Example 15 : ***(Optional)***

The instantaneous power delivered by an engine is given by

$P = (2t + 1)(t + 1)$ Watts

Find work done by the engine between

$t = 1s$ and $t = 2s$.

Solution :

$W = \int_1^2 P dt = \int_1^2 (2t + 1)(t + 1) \, dt = \int_1^2 (2t^2 + 2t + t + 1) \, dt = \int_1^2 (2t^2 + 3t + 1) \, dt$

$\Rightarrow \left[\frac{2}{3}t^3 + \frac{3}{2}t^2 + t\right]_1^2 = \left(\frac{2}{3} \times 8 + \frac{3}{2} \times 4 + 2\right) - \left(\frac{2}{3} + \frac{3}{2} + 1\right) = \frac{61}{6}$ J

Example 16 : ***(Optional)***

Force experienced by a particle in terms of its position $F = 3x^2 - 2x$. Calculate the work done in displacing the particle from *x* = 1 to *x* = 2.

Solution :

$$W = \int_1^2 F dx = \int_1^2 (3x^2 - 2x)\,dx = (x^3 - x^2)\Big|_1^2 = 4\text{ J}$$

Example 15 : ***(Optional)***

Accleration to particle travelling along *x* - axis is given by

$a = (2t + 1)\ m/s^2$

Find its velocity at t = 2 sec if velocity at t = 1 sec is 5 m/s

Solution :

$$V_{t=2} - V_{t=1} = \int_1^2 a dt$$

$$V_{t=2} - 5 = \int_1^2 (2t\ + 1)\,dt \quad V_{t=2} - 5 = t^2 + t\,|_1^2 \Rightarrow V_{t=2} - 5 = 6 - 2$$

$\therefore\ V_{t=2} = 9$ m/s

Example 16 : ***(Optional)***

In the same problem, find the position of the particle at *t* = 3. If initially position is x = 10.

Solution :

$$V_{t=t} - V_{t=1} = \int_1^t (2t + 1)\,dt$$

$$V_t - 5 = t^2 + t\,|_1^t$$

$$V_t - 5 = t^2 + t - 2$$

$$\boxed{V_t = t^2 + t + 3}$$

$$x_2 - x_1 = \int_{t_1}^{t_2} V dt$$

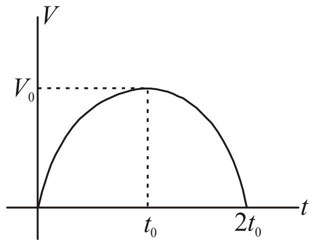
$$\Rightarrow \quad x_{t=3} - 10 = \int_0^3 t^2 + t + 3\ dt$$

$$\Rightarrow \quad x_{t=3} - 10 \quad \Rightarrow \quad \left[\frac{t^3}{3} + \frac{t^2}{2} + 3t\right]_0^3$$

$$x_{t=3} - 10 = 9 + \frac{9}{2} + 9$$

$$\Rightarrow \quad x_{t=3} = \frac{45}{2} + 10 \quad \Rightarrow \quad x_{t=3} = \frac{65}{2}\text{ m}$$

Example 17 :

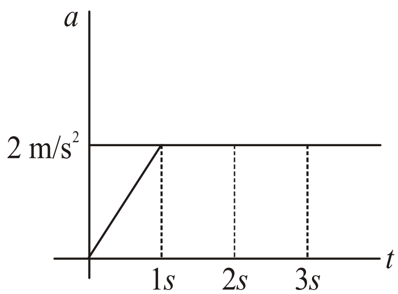


Find Acceleration at $t = t_{0,}$

Solution :

$$a = \frac{av}{dt} \text{ at } t = t_0, \frac{dv}{dt} = 0$$

Example 18 : ***(Optional)***



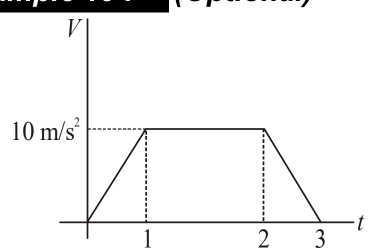
Find the velocity of the particle at t = 2*s*. If initial velocity is 5 m/s.

Solution :

$$V_{t=2} - V_{t=0} = \int_0^2 adt = \{\text{Area from 0 to 2}\} = \frac{1}{2}[1+2] \times 2$$

$$\Rightarrow \quad V_{t=2} = 3 + 5 = 8 \text{ m/s}$$

Example 19 : ***(Optional)***



The velocity time of particle traveling along v-axis is shown. Find

(i) Acceleration at t = 0.5 sec

(ii) Acceleration at t = 1.5 sec

(iii) Acceleration at t = 2.5 sec

(iv) At t = 3 sec, the particle is found at x = 50 m. What is its initial position.

Solution :

(i) 10 m/s^2 (ii) 0 m/s

(iii) – 10 m/s^2

(iv) $x_{t=3} - x_{t=0} = \int_0^3 V dt = \{\text{Area from 0 to 3}\} = \frac{1}{2} \times (1+3) \times 10$

$x_{t=3} - x_{t=0} = 20$

$50 - x_{t=0} = 20$

$x_{t=0} = 30 \text{ m}$

CHECK YOUR GRASP

- *If dy/dx = sinx, then find y.*
- *Find the integration of $3x^2$ with respect to x.*
- *What is difference between differntiation and integration.*
- *If* $\frac{dy}{dx} = -\sin x + \frac{1}{x}$, *then y =*

  (1) $-\cos x - \frac{1}{x^2}$ (2) $\cos x - \frac{1}{x^2}$

  (3) $\log x + \cos x + c$ (4) $\log x - \cos x + c$

- *If* $y = \frac{d}{dx} f(x)$ *then* $\int_{x_1}^{x_2} f(x) dx =$

  (1) *Change in y when x changes from x_1 to x_2*

  (2) *Area under the curve y versus (x) when x changes from x_1 to x_2*

  (3) *Change in x from x_1 to x_2*

  (4) *Function y*

VECTORS

INTRODUCTION

Mathematics is the language of Physics. Centrain physical quantities are completely described by numberial value alone and are added according to the ordinary rules of algebra. But the complete description of another set of physical quantities requires a numerical value (with units) as well as direction in space. It becomes easier to describe, understand and apply the physical principles, if one has a sound knowledge of vector algebra and calculus.

SCALARS

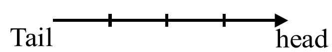
Certain physical quantities are completely described by a numerical value alone and are added according to the ordinary rules of algebra. Such quantities are called *scalars*.

e.g. If two bodies, one having of mass 5 kg and other having a mass of two kg are added together to make a composite system, the total mass of a system becomes 5kg + 2kg = 7kg.

VECTORS

The physical quantities which have magnitude and direction and which can be added according to the rules of vectors are called *vector quantities*.

Geometrically the vector is represented by a line of particular length, putting an arrow showing the direction of a vector.

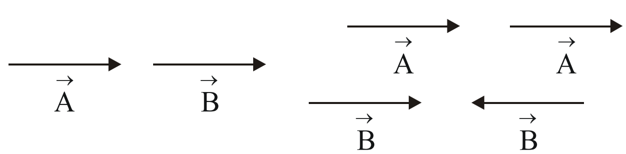


The front end is called the head and the rear end is called the tail. The vectors are denoted by putting an arrow over the symbols representing them. Thus we write $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. Sometimes a vector is represented by a single letter such as $\vec{v}, \vec{F}$ etc. Quite often in printed books the vectors are represented by bold face letters like ***AB***, ***BC***, **v**, ***f*** etc.

Note : *Electric current has both magnitude and direction but it does not add up according to the vector's rule. Hence it is not a vector quantity.*

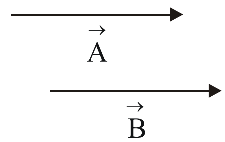
VARIOUS TYPES OF VECTORS

(a) **Collinear Vectors :** Two vectors having equal or unequal magnitude, which either act along the same line or along parallel lines in same direction or along the parallel lines in opposite direction, are called collinear vectors.

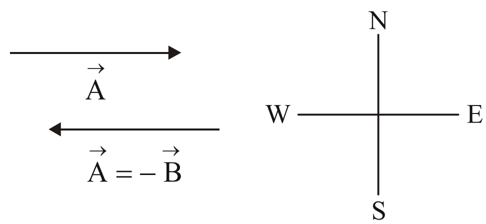


(b) Equal Vectors

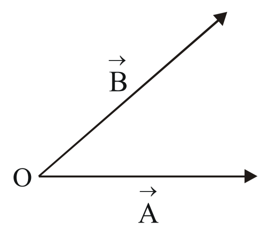
Two vectors are said to be equal, if they have the same magnitude and direction and if they represent same physical quantity.



(c) Negative Vector : The negative of a vector is defined as another vector having the same length representing the same physical quantity but drawn in opposite direction.



(d) Co-initial Vectors : Two vectors are said to be co-initial, if they have a common initial point. In figure, two vectors $\vec{A}$ and $\vec{B}$ have been drawn from the same point O. They are called co-initial vectors.



(e) Zero (or Null or Void or Empty) Vector ($\vec{0}$) : A vector whose magnitude is zero and direction is arbitary or unspecified is called zero vector.

e.g., If a particle starts moving from a point P, but comes back to P, then its displacement is zero vector.

(f) Unit Vector : A vector divided by its magnitude is called a unit vector along the direction of the vector. Obviously, the unit vector has unit magnitude and direction is the same as that of the given vector.

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

It is used for giving direction.

For e.g., a vector of magnitude 5 in direction of $\vec{a}$ is $5\hat{a}$

For e.g., a vector of magnitude 3 opposite to $\vec{a}$ is $-3\hat{a}$

CONCEPT BOOSTER

$\hat{i}$, $\hat{j}$, and $\hat{k}$ are known as unit vector corresponding to x, y and z axis respectively.

Example 1:

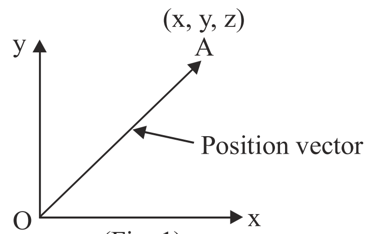
(i) A vector of 3 unit along x axis is $\vec{r} = 3\hat{i}$

(ii) A vector of magnitude 5 along –z axis is $\vec{r} = 5(-\hat{k}) = -5\hat{k}$

(g) Position Vector ($\overrightarrow{OA}$ or $\vec{r}$):

(a) It tells the straight line distance of the object from the origin starting point.

(b) It tells the direction of the object w.r.t. the origin.



(Fig. 1)

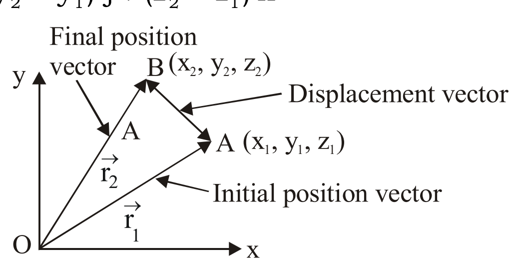
The magnitude of the position vector $\overrightarrow{OA}$ is represented by $|\overrightarrow{OA}|$ or $|\vec{r}|$ or r. Thus, $|\overrightarrow{OA}|$

or $|\vec{r}|$ or r = length of vector

$$\overrightarrow{OA} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(h) Displacement Vector ($\overrightarrow{AB}$ or $\vec{S}$) : A particle was situated at A at time t. The position co-ordinates of A (x_1, y_1, z_1). Suppose that at time t', the object reaches point B (as shown in figure 2). Then, $\overrightarrow{OB}$ is new position vector of the object or the position vector of the object in time t'. The arrow, whose tail is at A (initial position) and tip is at B (final position) is called the displacement vector of the object in time interval between t and t'. Both the position vector and the displacement vector described above are examples of vectors in two dimension.

$$\vec{S} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$



(Fig. 2)

(h) Axial Vectors : Vectors that represent rotational effects and are always along the axis of rotation in accordance with right hand screw rule are axial vectors along that direction.

e.g. angular velocity (w) ; angular acceleration (a) ; torque (t) etc.

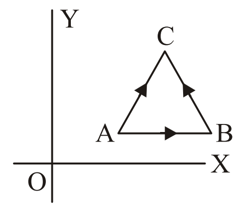
(i) Coplanar Vectors: A given number of vectors are said to be coplanar if they all lie in the same plane or parallel planes, otherwise they are said to be non-coplanar. Three or more vectors may or may not be coplanar.

(j) Resultant of two or more Vectors : It is a single vector which can produce the same effect which is produced by all those vectors acting together. Resultant of two or more vectors is obtained by adding those vectors.

ADDITION OF VECTORS

(i) Triangle law of addition of two vectors

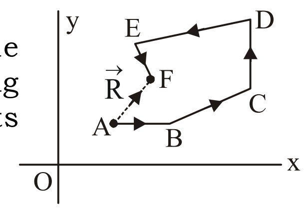
If two vectors can be represented fully (i.e., in magnitude as well as in direction) by the two sides of a triangle, taken in order, then the third side of the triangle taken in opposite order represents their resultant.



i.e. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(ii) Polygon Law of Addition of many Vectors

If a number of vectors can be represented fully by the successive sides of a polygon taken in order, then the closing side of that polygen, taken in opposite order, represents their resultant.

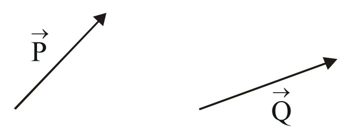


This means, in figure

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{AF}$

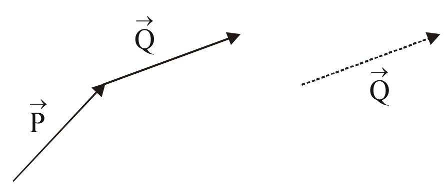
Addition of Vectors: Graphically

Suppose we want to add two given vectors $\vec{P}$ and $\vec{Q}$, i.e., we want to find out resultant of two given vectors $\vec{P}$ and $\vec{Q}$.

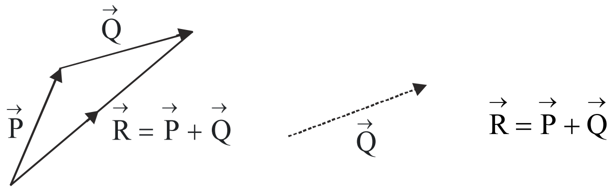


Step-wise procedure to do it

Step I: Shift $\vec{Q}$ such that its tail concides with the tip of $\vec{P}$.



Step II: Join the tail of $\vec{P}$ with the tip of thus shifted $\vec{Q}$. If we call this as $\vec{R}$, then

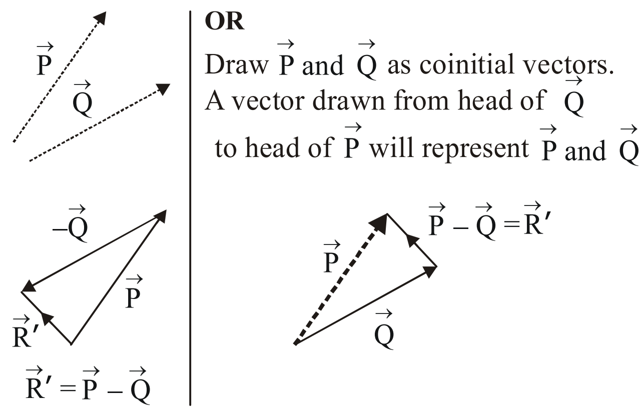


By triangle law of vector addition. $\vec{R}$ is called the resultant of $\vec{P}$ and $\vec{Q}$.

Subtraction using Triangle Law: If we have to find $\vec{P} - \vec{Q}$

Then we can write $\vec{P} - \vec{Q} = \vec{P} + (-\vec{Q})$

Thus, it is also a case a vector addition, it is the addition of $\vec{P}$ and the negative of $\vec{Q}$.



Example 2 :

Find the minimum number of nonparallel vector producing zero resultant.

Solution :

3

Example 3 :

Find the minimum number of non-coplanar vector producing zero resultant.

Solution :

4

Example 4 :

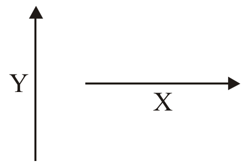
Which of the following group of magnitude will not produce zero resultant when taken in form of vector.

(1) 20, 3, 7, 4 (2) 20, 20, 20, 20 (3) 20, 3, 7, 10 (4) 20, 3, 7, 12

Solution :

(1)

Example 5 :

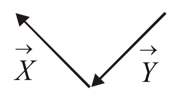


Two vectors $\vec{X}$ and $\vec{Y}$ are shown. Match the following:

(i) ↖	(a) $\vec{X} + \vec{Y}$
(ii) ↘	(b) $-(\vec{X} + \vec{Y})$
(iii) ↙	(c) $\vec{X} - \vec{Y}$
(iv) ↗	(d) $\vec{Y} - \vec{X}$

Ans. (i) - (d), (ii) - (c), (iii) - (b), (iv) - a

Example 6 :



The vectors $\vec{X}$ and $\vec{Y}$ are as shown. Match the following :

(i) ↓	(a) $\vec{X} + \vec{Y}$
(ii) ↑	(b) $-(\vec{X} + \vec{Y})$
(iii) →	(c) $\vec{X} - \vec{Y}$
(iv) ←	(d) $\vec{Y} - \vec{X}$

Solution :

(i) (d) (ii) (c) (iii) (b) (iv) (a)

CHECK YOUR GRASP

- ☞ *We can order events in time and there is a sense of time, distinguishing past, present and future. Is, therefore, time a vector.*
- ☞ *Explain why current is not a vector though it appears to possess a direction*
- ☞ *Discuss whether or not angular displacement is a vector*

EVALUATE YOURSELF - 1

1. What is the maximum number of rectangular components into which a vector can be split in its own plane?

(1) 2 (2) 3 (3) 4 (4) Infinite

2. The minimum number of vectors of equal magnitude required to produce a zero resultant is

(1) 2 (2) 3 (3) 4 (4) More than 4

3. A vector does not change if

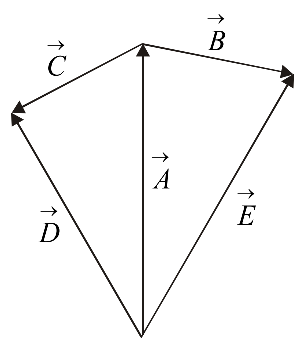
(1) It is rotated through an arbitrary angle

(2) It is multiplied by an arbitrary scalar

(3) It is cross multiplied by a unit vector

(4) It is slid parallel to it self

4. In the figure, $\vec{E} - \vec{D} + \vec{C}$ equals



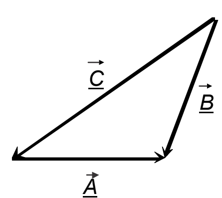
(1) $\vec{A}$ (2) $-\vec{A}$ (3) $\vec{B}$ (4) $-\vec{B}$

5. Two vectors P and Q are in a plane but the vector R is not in their plane. In such a case $\vec{P} + \vec{Q} + \vec{R}$

(1) Can be zero (2) Can not be zero

(3) Lies in the same plane as P & R (4) Lies in the same plane as R & Q

6. For the fig.



(1) $\vec{A} + \vec{B} = \vec{C}$

(2) $\vec{B} + \vec{C} = \vec{A}$

(3) $\vec{C} + \vec{A} = \vec{B}$

(4) $\vec{A} + \vec{B} + \vec{C} = 0$

7. Let $\vec{C} = \vec{A} + \vec{B}$ then

(1) $|\vec{C}|$ is always greater then $|\vec{A}|$

(2) It is possible to have $|\vec{C}|<|\vec{A}|$ and $|\vec{C}|<|\vec{B}|$

(3) C is always equal to $A + B$

(4) C is never equal to $A + B$

8. A person moves 30m north, then 40m east and then 50m south-west. His displacement from the original position is nearly

(1) 120 m (2) 100 m (3) 70 m (4) zero

ANSWER

1. (4)	2. (1)	3. (4)	4. (3)	5. (2)
6. (3)	7. (2)	8. (4)		

(III) PARALLELOGRAM METHOD

The same rule may be stated in a slightly different way. We draw the vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding.

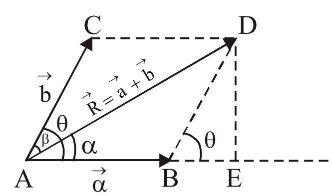
Taking these two as the adjacent sides we complete the parallelogram.

The diagonal through the common tails gives the sum of the two vectors.

Thus, in figure $\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$.

Suppose the angle between $\vec{a}$ and $\vec{b}$ is θ. It is easy to see from the figure that

$AD^2 = (AB+BE)^2+(DE)^2$



$R^2 = (a+b\cos\theta)^2+(b\sin\theta)^2$

$= a^2+2ab\cos\theta+b^2$

Thus, the magnitude of $\vec{a}+\vec{b}$ is,

$R = \sqrt{a^2+b^2+2ab\cos\theta}$

Its angle with $\vec{a}$ is α

where, $\tan\alpha = \dfrac{DE}{AE}=\dfrac{b\sin\theta}{a+b\cos\theta}$ and its angle β with $\vec{b}$

$$\tan\beta=\frac{a\sin\theta}{b+a\cos\theta}$$

Let us find the resultant of the two vectors in the following special cases:

(i) When $\theta=0°$ i.e., when the two vectors act along the same direction.

(R is maximum in this case)

R $= \sqrt{a^2 + b^2 + 2ab\cos 0^\circ} = \sqrt{a^2 + b^2 + 2ab(1)}$

$= a + b$

Also,

$$\tan\alpha = \frac{b\sin 0^0}{a + b\cos 0^0} = \frac{b(0)}{a + b(1)} = 0 \quad \Rightarrow \quad \alpha = 0^\circ$$

Therefore, when $\theta = 0^0$, the resultant has magnitude equal to the sum of the magnitude of two vectors and acts along the direction of $\vec{a}$ and $\vec{b}$. Also $\beta = 0^\circ$

(ii) **When $\theta = 90^\circ$, i.e., when the two vectors act at right angles to each other.**

$R^2 = \sqrt{a^2 + b^2 + 2ab\cos 90^\circ} = \sqrt{a^2 + b^2 + 2ab(0)}$

or $\sqrt{a^2 + b^2}$

Also, $\tan\alpha = \dfrac{b\sin 90^\circ}{a + b\cos 90^\circ} = \dfrac{b(1)}{a + b(0)} = \dfrac{b}{a}$ and $\tan\beta = a/b$

(iii) **When $\theta = 180^\circ$ i.e., when the two vectors act along opposite directions. (R is minimum in this case).**

We have,

$R = \sqrt{a^2 + b^2 + 2ab\cos 180^\circ} = \sqrt{a^2 + b^2 + 2ab(-1)}$

or $R = a - b$ or $b - a$

Also,

$$\tan\alpha = \frac{b\sin 180^\circ}{a + b\cos 180^\circ} = \frac{b(0)}{a + b(-1)} = 0$$

or $\alpha = 0^\circ$ or 180° and $\beta = 180^\circ$ or 0°

Therefore, when $\theta = 180^\circ$, the resultant has magnitude equal to the difference of their magnitudes and acts along the direction of the bigger of the two vectors.

It may be pointed out that the magnitude of the resultant of two vectors is maximum, when they act along the same direction and minimum, when they act along the opposite directions.

MULTIPLICATION OF A VECTOR BY A NUMBER

Suppose $\vec{a}$ is vector of magnitude a and k is a number. Let us define the vector $\vec{b} = k\vec{a}$. Then $\vec{b}$ is as a vector of magnitude $|ka|$. If k is positive, the direction of the vector $\vec{b}$ is same as that of $\vec{a}$. If k is negative, the direction of $\vec{b}$ is opposite to $\vec{a}$. In particular, multiplication by (-1) just inverts the direction of the vector. So, the vectors $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite in directions.

Note : If $\vec{a}$ is a vector of magnitude $\vec{a}$ and $\hat{u}$ is a vector of unit magnitude in the direction of $\vec{a}$, we can write $\vec{a} = a\hat{u}$.

SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$ invert the direction of $\vec{b}$ and add to $\vec{a}$.

$\therefore$ $$R' = |\vec{a} - \vec{b}| = \sqrt{a^2 + b^2 + 2ab\cos(\pi - \theta)}$$

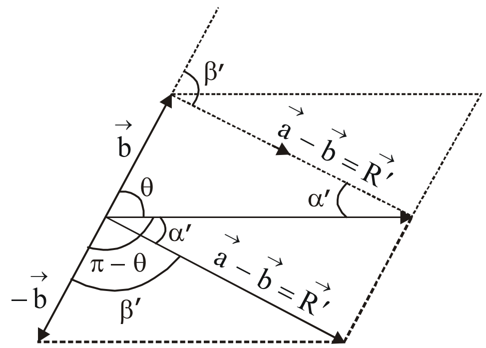
or $$R' = |\vec{a} - \vec{b}| = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

$\alpha' =$ angle between $\vec{a}$ and $\vec{a} - \vec{b}$

and $\beta' =$ angle between $\vec{b}$ and $\vec{a} - \vec{b}$

Also $$\tan\alpha' = \frac{b\sin(\pi - \theta)}{a + b\cos(\pi - \theta)} = \frac{b\sin\theta}{a - b\cos\theta}$$

or $$\frac{-b\sin\theta}{a - b\sin\theta}$$



CONCEPT BOOSTER

If $|\vec{a}| = |\vec{b}| = x$

(1) $R = |\vec{a} + \vec{b}| = \sqrt{x^2 + x^2 + 2x^2\cos\theta} = 2x\cos(\theta/2)$

(ii) $R' = |\vec{a} - \vec{b}| = \sqrt{x^2 + x^2 - 2x^2\cos\theta} = 2x\sin(\theta/2)$

(iii) $\alpha = \beta = (\theta/2)$

(iv) Parallelogram formed by $\vec{a}$ and $\vec{b}$ will become a rhombus, whose diagonals bisect each other perpendicularly. In this case

$\therefore$ $(\vec{a} + \vec{b})$ is $\perp$ $(\vec{a} - \vec{b})$

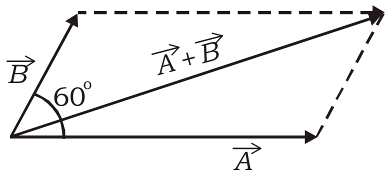
Example 7 :

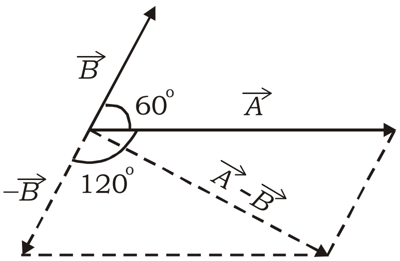
Two vectors of equal magnitude 5 unit have an angle 60° between them. Find the magnitude of (a) of the sum of the vectors and (b) the difference of the vectors.

Solution :

Figure shows the construction of the sum $\vec{A}+\vec{B}$ and the difference $\vec{A}-\vec{B}$.

(a) $\vec{A}+\vec{B}$ is the sum of $\vec{A}$ and $\vec{B}$. Both have a magnitude of 5 unit and the angle between them is 60°. Thus the magnitude of the sum is





$$\left|\vec{A}+\vec{B}\right| = \sqrt{5^2+5^2+2\times5\times5\cos60^\circ} = 5\sqrt{3}\ \text{unit}$$

(b) $\vec{A}-\vec{B}$ is the sum of $\vec{A}$ and $(-\vec{B})$. As shown in the figure, the angle between $\vec{A}$ and $(-\vec{B})$ is 120°. The magnitudes of both $\vec{A}$ and $(-\vec{B})$ is 5 unit, so,

$$\left|\vec{A}-\vec{B}\right| = \sqrt{5^2+5^2+2\times5\times5\cos120^\circ} = 5\ \text{unit}$$

Example 8:

The forces of 10 N and 6 N act upon a body. The directions of the forces are unknown. The resultant force on the body may lie between which limits ?

Solution :

Let θ is the angle between the two forces P = 10 N and Q = 6 N. Then, the resultant force is given by

$$R = \sqrt{10^2 + 5^2 + 2\times10\times6\cos\theta}$$

The minimum value of cos q is –1, while, the maximum values is +1.

Thus, $$R_{min} = \sqrt{10^2+6^2+2\times10\times6\times(-1)} = \sqrt{(10-6)^2} = 4N$$

$$R_{max} = \sqrt{10^2+6^2+2\times10\times6\times(+1)} = \sqrt{(10+6)^2} = 16N$$

Thus, the resultant of the given two forces lies between 4 N and 16 N.

Note : The maximum and minimum magnitude of the resultant of two vectors is always $|\vec{P}|+|\vec{Q}|$ and $|\vec{P}|-|\vec{Q}|$. So, in this case, it will be 10 + 6 = 16 N and 10 – 6 = 4 N.

Example 9:

The vector sum of two vectors $\vec{P}$ and $\vec{Q}$ is R. If vector $\vec{Q}$ is reversed, the resultant becomes $\vec{S}$. Then, prove that $R^2 + S^2 = 2(P^2 + Q^2)$

Solution :

Let θ be the angle between vectors $\vec{P}$ and $\vec{Q}$.

Then, $R^2 = P^2 + Q^2 + PQ\cos\theta$ ... (i)

When vector $\vec{Q}$ is reversed, angle between the vector $\vec{P}$ and $-\vec{Q}$ will become $180° - \theta$.

Thus, $S^2 = P^2 + Q^2 + 2PQ\cos(180° - \theta)$

$= P^2 + Q^2 + 2PQ \times (-\cos\theta)$

or $S^2 = P^2 + Q^2 - 2PQ\cos\theta$ ... (ii)

Adding the equestions (i) and (ii), we have

$$R^2 + S^2 = 2(P^2 + Q^2)$$

Example 10 :

Two vectors of magnitude 6 and 14 give a resultant R, then

(a) $-8 < R < 20$ (b) $8 < R < 20$ (c) $-8 \le R \le 20$ (d) $8 \le R \le 20$

Solution :

(d)

Example 11 :

Two vectors of magnitude x when added gives a vector of magnitude of x. Find the angle between two vectors which are added.

Solution :

$R = 2x\cos\theta/2 = x$

$\therefore \cos\theta/2 = 1/2$

$\therefore \theta/2 = 60°$

$\therefore \theta = 120°$

Example 12 :

Two vectors of magnitude x when subtracted gives a vector of magnitude of x. Find the angle between two vectors which are subtracted.

Solution :

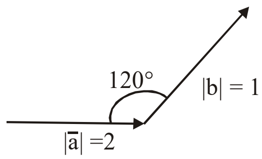
$R = 2x\sin\theta/2 = x$

$\therefore \sin\theta/2 = 1/2$

$\therefore \theta/2 = 30°$

$\therefore \theta = 60°$

Example 13 :



Find the resultant and the angle made by resultant with b.

Solution :

$$R^2 = \sqrt{2^2 + 1^2 + 2 \times 2 \times 1 \cos 60^\circ} = \sqrt{4 + 1 + 4 \times \frac{1}{2}}$$

$$R = \sqrt{5 + 2}$$

$$R = \sqrt{7}$$

$$\tan \beta = \frac{a \sin \theta}{b + a \cos \theta}$$

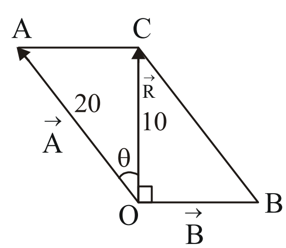
$$\beta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Example 14 :

The resultant of two vectors A and B where B > A is of magnitude 10 and perpendicular to B. If the magnitude A is 20. Find the magnitude of B and angle between A and B.

Solution :

In Δ AOC



$$\cos\theta = \frac{10}{20} = \frac{1}{2}$$

$\therefore$ $\theta = 60^\circ$

$\therefore$ Angle between $\vec{A}$ and $\vec{B} = 150^\circ$

$$|\vec{B}| = \sqrt{20^2 - 10^2} = \sqrt{400 - 100} = \sqrt{300}$$

CHECK YOUR GRASP

☞ *Is vector addition commutative?*

☞ *Is vector addition associative?*

EVALUATE YOURSELF - 2

1. Vector sum of two forces of 10N and 6N cannot be:

(1) 4 N (2) 8 N (3) 12 N (4) 2 N

2. At what angle the two vectors of magnitudes (A + B) and (A – B) must act, so that the resultant is $\sqrt{A^2+B^2}$?

(1) $\cos^{-1}\frac{A^2-B^2}{A^2+B^2}$ (2) $\cos^{-1}\frac{A^2+B^2}{B^2-A^2}$

(3) $\cos^{-1}\frac{A^2-B^2}{2(A^2+B^2)}$ (4) $\cos^{-1}\frac{A^2+B^2}{2(B^2-A^2)}$

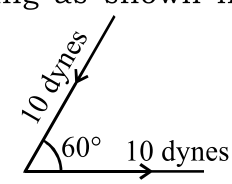
3. Resultant of the two vector $\vec{F}_1$ and $\vec{F}_2$ is of magnitude P. If $\vec{F}_2$ is reversed, then resultant is of magnitude Q. What is the value of $P^2 + Q^2$?

(1) $F_1^2+F_2^2$ (2) $F_1^2-F_2^2$ (3) $2(F_1^2-F_2^2)$ (4) $2(F_1^2+F_2^2)$

4. The resultant of two forces acting at an angle of 150° is 10kg wt. and is perpendicular to one of the forces. The other force is ....

(1) $10\sqrt{3}$ kg.wt. (2) $20\sqrt{3}$ kg.wt. (3) 20 kg.wt. (4) $20/\sqrt{3}$ kg.wt.

5. Two forces each numerically equal to 10 dynes are acting as shown in the following figure, then the resultant is



(1) 10 dynes (2) 20 dynes

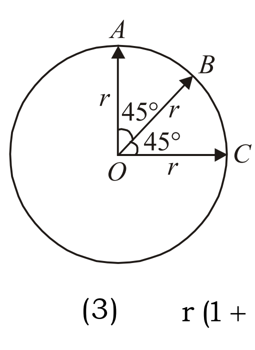
(3) $10\sqrt{3}$ dynes (4) 5 dynes

6. Given that $\vec{R}=\vec{P}+\vec{Q}$. And $\vec{R}$ makes angle α with $\vec{P}$ and β with $\vec{Q}$. Which of the following relations is correct?

(1) α cannot be less than β (2) $\alpha<\beta$ if P < Q

(3) $\alpha<\beta$ if P > Q (4) $\alpha<\beta$ if P = Q

7. The resultant of the three vectors $\overrightarrow{OA}, \overrightarrow{OB},$ and $\overrightarrow{OC}$ shown in figure.



(1) r (2) 2r (3) $r(1+\sqrt{2})$ (4) $r(\sqrt{2}-1)$

8. At what angle should the two forces 2P and $\sqrt{2}$P act so that the resultant force is P$\sqrt{10}$

(1) 45° (2) 60° (3) 90° (4) 120°

9. If $|\vec{A}-\vec{B}|=|\vec{A}|-|\vec{B}|$, the angle between $\vec{A}$ and $\vec{B}$ is

(1) 60° (2) 0° (3) 120° (4) 90°

ANSWERS

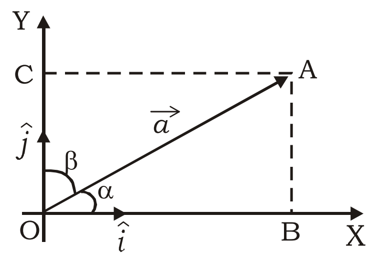
1. (4)	2. (4)	3. (4)	4. (3)	5. (1)
6. (3)	7. (3)	8. (1)	9. (2)	

RESOLUTION OF VECTORS

A vector $\vec{a}=\overline{OA}$ in the *X-Y* plane drawn from the origin *O*. The vector makes an angle a with the X-axis and b with the Y-axis. Draw perpendiculars *AB* and AC from *A* to the *X* and *Y* axis respectively. The length *OB* is called the projection of $\overline{OA}$ on X-axis. According to rule of vector addition

$$\vec{a}=\overrightarrow{OA}=\overrightarrow{OB}+\overrightarrow{OC}$$

Thus, we have resolved the vector $\vec{a}$ into two parts, one along *OX* and the other along *OY*. The magnitude of the part along *OX* is $OB=a\cos\alpha$ and the magnitude of the part along *OY* is $OC=a\cos\beta$. If $\hat{i}$ and $\hat{j}$ denote vectors of unit magnitude along *OX* and *OY* respectively. We get



$\overrightarrow{OB}=a\cos\alpha\hat{i}$ and $\overrightarrow{OC}=a\sin\alpha\hat{j}$

So that $\vec{a}=a\cos\alpha\hat{i}+\alpha\sin\alpha\hat{j}$

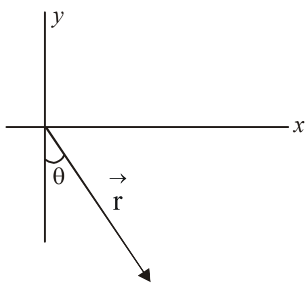
i.e., $\vec{a}=a_x\hat{i}+a_y\hat{j}$

where, $a_x=a\cos\alpha$ and $a_y=a\sin\alpha$

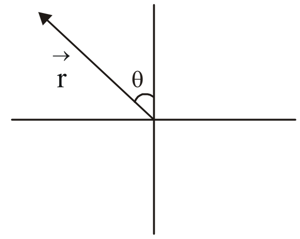
where, magnitude of $|\vec{\mathbf{a}}|=\sqrt{\mathbf{a}_x^2+\mathbf{a}_y^2}$

where, $\tan\alpha=\frac{\mathbf{a}_y}{\mathbf{a}_x}$ **and** $\tan\beta=\frac{\mathbf{a}_x}{\mathbf{a}_y}$

e.g.



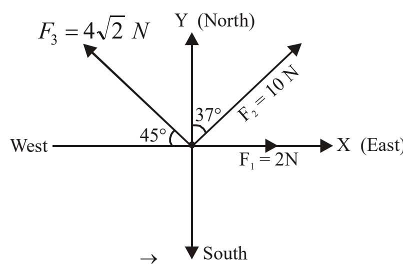
e.g.



$\vec{r} = r(\sin\theta)\,\hat{i} - (r\cos\theta)\,\hat{j}$

$\vec{r} = -r(\sin\theta)\,\hat{i} + (r\cos\theta)\,\hat{j}$

Example 15 :



(Given sin 37° = cos 53° = 3/5, cos 37° = sin 53° = 4/5)

Find the resultant force and its direction.

Solution :

$\vec{F}_1 = 2\hat{i}$

$F_2 = 10\cos 37\hat{j} + 10\sin 37\hat{i} = 6\hat{i} + 8\hat{j}$

$\vec{F}_3 = -4\hat{i} + 4\hat{j}$

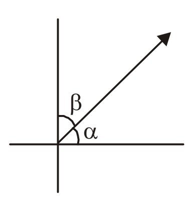
$\vec{F}_4 = -10\hat{j}$

$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 4\hat{i} + 2\hat{j}$

$F = \sqrt{4^2 + 2^2} = \sqrt{20}\ \text{N}$

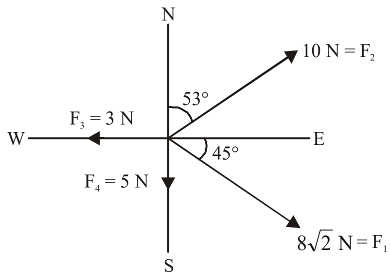
$\alpha = \tan^{-1}(F_y / F_x) = \tan^{-1}(1/2)$

North of East $\beta = \tan^{-1}\left(\dfrac{F_x}{F_y}\right)$



$= \tan^{-1}(2)$ East of North

Example 16 :



Find the resultant force and its magnitude.

Solution :

$\vec{F}_1 = 8\hat{i} - 8\hat{j} \quad \vec{F}_2 = 8\hat{i} + 6\hat{j}$

$\vec{F}_3 = -3\hat{i} \quad \vec{F}_4 = -5\hat{j}$

Adding all

$\therefore \quad \vec{F} = 13\hat{i} - 7\hat{j} \qquad \alpha = \tan^{-1}\left(\frac{7}{13}\right)$ South of East

Example 17 :

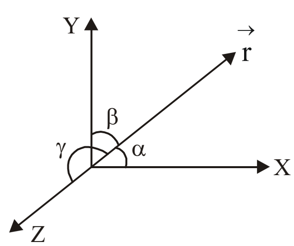
A force of 10.5 N acts on a particle along a direction making an angle of 37° with the vertical. Find the component of the force in the vertical direction.

Solution :

The component of the force in the vertical direction will be F_1

$$= F\cos\theta = (10.5\text{N})(\cos 37°) = (10.5\text{N})\frac{4}{5} = 8.4 \text{ N}$$

THREE DIMENSIONAL VECTORS



$\vec{r} = r\cos\alpha\,\hat{i} + r\cos\beta\hat{j} + r\cos\gamma\hat{k}$

$= r(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$

Direction cosines

$l = \cos\alpha = \frac{r_x}{r}; \qquad m = \cos\beta = \frac{r_y}{r}; \qquad n = \cos\gamma = \frac{r_z}{r} \qquad$ where, $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$

$\therefore \quad \hat{r} = l\hat{i} + m\hat{j} + n\hat{k} \qquad$ Also, $l^2 + m^2 + n^2 = 1$

Example 18 :

Find the direction cosines of vector.

$\vec{r} = 3\vec{a} - 2\vec{b} + \vec{c}$ where, $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$; $\vec{c} = -\hat{i} - \hat{j} + \hat{k}$

Solution :

$$\vec{r} = 3(\hat{i} - 2\hat{j} + \hat{k}) - 2(2\hat{i} + \hat{j} + \hat{k}) + (-\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = -2\hat{i} - 9\hat{j} + 2\hat{k}$$

$$|\vec{r}| = \sqrt{89}; \quad l = \frac{-2}{\sqrt{89}}$$

$$m = \frac{-9}{\sqrt{89}}; \quad n = \frac{2}{\sqrt{89}}$$

Example 19 :

Three dimensional vector of magnitude 9 has direction cosines $\left(\frac{1}{3}, \frac{-2}{3}, n\right)$. Find all such vectors.

Solution :

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + n^2 = 1 = \frac{1}{9} + \frac{4}{9} + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{9} - \frac{4}{9} = \frac{4}{9}$$

$$n = \pm\frac{2}{3}$$

$$\vec{r} = r(l\hat{i} + n\hat{j} + n\hat{k})$$

$$\vec{r} = 3[\hat{i} - 2\hat{j} \pm 2\hat{k}]$$

CHECK YOUR GRASP

☞ *Can a vector be zero if any of its components is not zero?*

EVALUATE YOURSELF - 3

1. What vector must be added to the sum of two vectors $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + 2\hat{k}$ so that the resultant is a unit vector along Z axis

(1) $5\hat{i} + \hat{k}$ (2) $-5\hat{i} + 3\hat{j} - 4\hat{k}$ (3) $3\hat{j} + \hat{k}$ (4) $-3\hat{j} + \hat{k}$

2. If $\vec{A} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{B} = 3\hat{i} + 6\hat{j} + 2\hat{k}$ then the vector in the direction of $\vec{A}$ and having magnitude as $|\vec{B}|$, is

(1) $7(\hat{i} + 2\hat{j} + 2\hat{k})$ (2) $\frac{3}{7}(\hat{i} + 2\hat{j} + 2\hat{k})$ (3) $\frac{7}{9}(\hat{i} + 2\hat{j} + 2\hat{k})$ (4) $\frac{7}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

3. $\vec{P} + \vec{Q}$ is a unit vector along y axis. If $\vec{P} = 2\hat{i} - \hat{j} + 3\hat{k}$, then $\vec{Q}$ is

(1) $-2\hat{i} + \hat{j} - 3\hat{k}$ (2) $-2\hat{i} + 2\hat{j} - 3\hat{k}$ (3) $-3\hat{i} + \hat{j} - 3\hat{k}$ (4) $2\hat{i} + \hat{j} + 3\hat{k}$

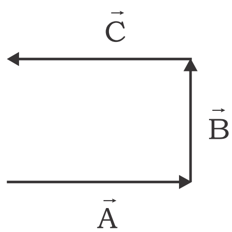
4. Which of the following is a unit vector ?

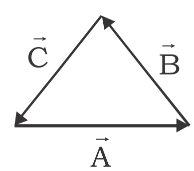
(1) $\hat{i} + \hat{j}$ (2) $\cos\theta\ \hat{i} - \sin\theta\ \hat{j}$ (3) $\sin\theta\ \hat{i} + 2\cos\theta\hat{j}$ (4) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j})$

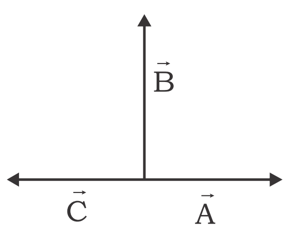
5. One of two rectangular components of a force is 20 N and it makes an angle of 60^o with the force. Then the magnitude of other component will be

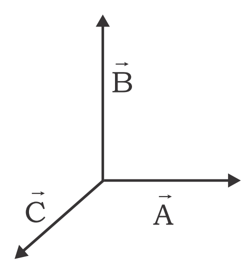
(1) $\frac{20}{\sqrt{3}}$ N (2) $20\sqrt{3}$ N (3) 40 N (4) zero

6. If $\vec{A}, \vec{B}, \vec{C}$ represent unit vectors in each case, which vector combination(s) result in a unit vector ?

(A)


(B)


(C)


(D)


(1) A, B, C (2) A, C (3) A, B, D (4) D

7. The resultant of $\vec{P}$ and $\vec{Q}$ is perpendicular to $\vec{P}$. What is the angle between and $\vec{P}$ and $\vec{Q}$

(1) $\cos^{-1}(P/Q)$ (2) $\cos^{-1}(-P/Q)$ (3) $\sin^{-1}(P/Q)$ (4) $\sin^{-1}(-P/Q)$

8. The value of a unit vector in the direction of vector $A = 5\hat{i} - 12\hat{j}$, is

(1) $\hat{i}$ (2) $\hat{j}$ (3) $(\hat{i} + \hat{j})/13$ (4) $(5\hat{i} - 12\hat{j})/13$

9. The expression $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ is a

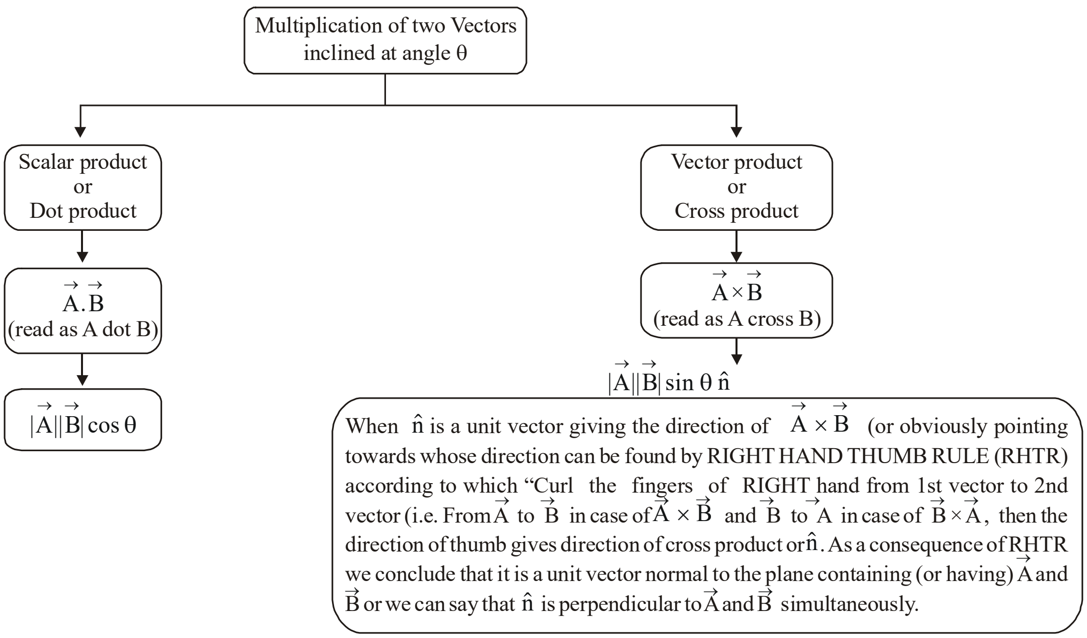
(1) Unit vector (2) Null vector

(3) Vector of magnitude (4) Scalar

ANSWER

1. (2)	2. (4)	3. (2)	4. (2)	5. (2)
6. (2)	7. (2)	8. (4)	9. (1)	

VECTOR MULTIPLICATION



DOT PRODUCT

Mathematically,

$$\vec{A}.\vec{B} = AB\cos\theta$$

PROPERTIES OF DOT PRODUCT

(a) Dot product is commutative in nature

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

(b) Dot product is distributive with respect to sum

$$\vec{A}.(\vec{B} \pm \vec{C}) = \vec{A}\vec{B} \pm \vec{A}.\vec{C}$$

(c) If θ = 0°, i.e. vectors are parallel then

$$\vec{A}.\vec{B} = AB\cos 0° = AB$$

i.e., when the vectors are parallel $\vec{A}.\vec{B}$ is just the simple product of the magnitude of $\vec{A}$ and $\vec{B}$.

(d) Dot product of vector with itself is equal to the square of the magnitude of the vector.

$\vec{A}.\vec{A} = (A)(A)\cos 0 \Rightarrow \vec{A}.\vec{A} = A^2$

(e) If θ = 180° i.e., vectors are antiparallel, then

$\vec{A}.\vec{B} = AB(-1)$ $\{\because (\cos 180° = -1)\}$

$\vec{A}.\vec{B} = -AB$

i.e., if two vectors are antiparallel then their dot product equals the negative of simple product of magnitudes of vectors.

(f) If θ = 90° i.e., vectors are perpendicular.

$\vec{A}.\vec{B} = AB(0)$ $\vec{A}.\vec{B} = 0$ i.e.,

Vectors perpendicular $\Leftrightarrow$ Dot product = 0

(g) $\hat{i}.\hat{i} = |\hat{i}||\hat{i}|\cos 0°$

Therefore, in general,

$\hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1$

(h) $\hat{i}.\hat{j} = |\hat{i}||\hat{i}|\cos 90° \Rightarrow \hat{i}.\hat{j} = (1)(1)(0)$

Therefore, in general

$\hat{i}.\hat{j} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0$ or $\hat{j}.\hat{i} = 0, \hat{k}.\hat{j} = 0, \hat{i}.\hat{k} = 0$

(i) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ are any two vectors, then

$\vec{A}.\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$\Rightarrow \vec{A}.\vec{B} = A_x\hat{i}.(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) +$

$A_y\hat{j}.(B_x\hat{i} + B_y\hat{j} + \hat{k}) + A_z\hat{k}.(B_x i + B_y\hat{j} + B_z\hat{k})$

$\therefore \vec{A}.\vec{B} = A_x B_x + A_yB_y + A_zB_z$

(j) Since, $\vec{A}.\vec{B} = AB\cos\theta$ and $-1 \le \cos\theta \le 1$

$\Rightarrow \vec{A}.\vec{B}$ is maximum at $\cos\theta = 1$ (i.e., $\theta = 0°$)

$\vec{A}.\vec{B}$ is minimum at $\cos\theta = -1$ (i.e., $\theta = 180°$)

$\Rightarrow (\vec{A}.\vec{B})_{min} \le \vec{A}.\vec{B} \le (\vec{A}.\vec{B})_{max}$

$\Rightarrow \quad -AB \le \vec{A}.\vec{B} \le AB$

(k) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B}$

$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and θ is angle between $\vec{A}$ and $\vec{B}$.

Then, $\cos\theta = \dfrac{\vec{A}.\vec{B}}{AB}$ where $\vec{A}.\vec{B} = AB\cos\theta$.

CONCEPT BOOSTER

Angle between two vectors

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ is

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

If $\vec{A} = A_z\hat{i} + A_y\hat{j} + A_z\hat{k}$, has l_1, m_1, n_1 as direction cosines

If $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, has l_2, m_2, n_2 as direction cosines

Also, $\cos\theta = \dfrac{A_xB_x + A_yB_y + A_zB_z}{AB}$

$$\Rightarrow \cos\theta = \frac{A_xB_x}{AB} + \frac{A_yB_y}{AB} + \frac{A_zB_z}{AB}$$

$$\Rightarrow \cos\theta = \left(\frac{A_x}{A}\right)\left(\frac{B_x}{B}\right) + \left(\frac{A_y}{A}\right)\left(\frac{B_y}{B}\right) + \left(\frac{A_z}{A}\right)\left(\frac{B_z}{B}\right)$$

$= l_1l_2 + m_1m_2 + n_1n_2$

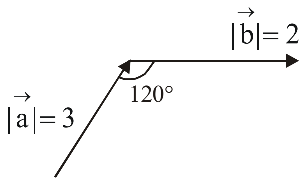
(a) If $\langle l_1\ m_1\ n_1 \rangle$ and $\langle l_2\ m_2\ n_2 \rangle$ are direction cosines of $\vec{A}$ and $\vec{B}$, then $\cos\theta = l_1\,l_2 + m_1\,m_2 + n_1\,n_2$.

(b) It two vectors are perpendicular then

$l_1\,l_2 + m_1\,m_2 + n_1\,n_2 = 0$.

Example 20 :

Find $\vec{a}\,.\,\vec{b}$, for the vectors shown in diagram.



Solution :

$\vec{a}\cdot\vec{b} = 3 \times 2 \cos 60° = 3$

Example 21 :

Find the value of α for $\vec{a} = 8\hat{i} - \alpha\hat{j} + 2\hat{k}$ is $\perp$ $\vec{b} = 2\hat{i} + \alpha\hat{j} + 8\hat{k}$

Solution :

$\vec{a}\cdot\vec{b} = 0$

$16 - \alpha^2 + 16 = 0$

$\alpha^2 = 32$

$\alpha = \pm 4\sqrt{2}$

Example 22 :

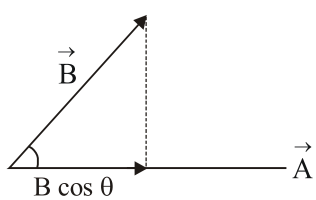
Find angle between $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

Solution :

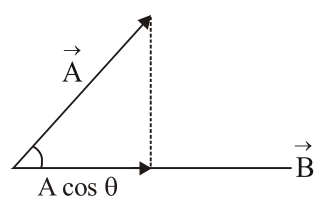
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{12-4-3}{21} = \frac{12-7}{21}$$

$$\theta = \cos^{-1}\frac{5}{21}$$

GEOMETRICAL INTERPRETATION



OR



$\vec{A}\cdot\vec{B} = A(B\cos\theta)$

$\vec{B}\cdot\vec{A} = B\,(A\cos\theta)$

$\Rightarrow$ $\vec{A}\cdot\vec{B} = A \times$ (projection of B in the direction of A)

$\Rightarrow \quad \vec{B}\,.\,\vec{A} = B \times$ (Projection of A in the direction of B)

$$\Rightarrow \quad B\cos\theta = \frac{\vec{A}.\vec{B}}{A} \quad \Rightarrow \quad A\cos\theta = \frac{\vec{A}.\vec{B}}{B}$$

In vector form

$$(B\cos\theta)\hat{A} = \left(\frac{\vec{A}.\vec{B}}{A}\right)\hat{A} = \left(\frac{\vec{A}.\vec{B}}{A^2}\right)\vec{A}$$

$$(A\cos\theta)\hat{B} = \left(\frac{\vec{A}.\vec{B}}{B}\right)\hat{B} = \left(\frac{\vec{A}.\vec{B}}{B^2}\right)\vec{B}$$

Example 23 :

Find projection of $\bar{a}$ on $\bar{b}$ if

$\vec{a} = 3\hat{i} + 2\hat{j}$

$\vec{b} = \hat{i} + \hat{j}$

Solution :

$$(a\cos\theta)\hat{b} = \left(\frac{\vec{a}\,.\,\vec{b}}{b}\right)\hat{b} = \frac{5}{\sqrt{2}}\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = \frac{5}{2}\left(\hat{i}+\hat{j}\right)$$

Example 24:

Find projection of $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ on $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$

Solution :

$$(b\cos\theta)\,\hat{a} = \left(\frac{2-2-1}{3}\right)\left(\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}\right)$$

$$= \frac{-1}{9}\left(2\hat{i}+2\hat{j}-\hat{k}\right) = \frac{-2\hat{i}-2\hat{j}+\hat{k}}{9}$$

Example 25 :

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 2$, $|\vec{b}| = 3$, $|\vec{c}| = 4$

Find angle between $\vec{b}$ and $\vec{c}$.

Solution :

$$(\vec{a})^2 = (\vec{c} - \vec{b})^2 \quad \Rightarrow \quad a^2 = c^2 + b^2 - 2bc\cos\theta$$

$$\Rightarrow\ 2bc\cos\theta = c^2 + b^2 - a^2$$

$$\cos\theta = \frac{c^2 + b^2 - a^2}{2bc}$$

$$\cos\theta = \frac{(4)^2 + (3)^2 - (2)^2}{2 \times 3 \times 4}$$

$$\cos\theta = \frac{16 + 9 - 4}{24} = \frac{7}{8} \quad \Rightarrow \quad \theta = \cos^{-1}(7/8)$$

PHYSICAL INTERPRETATION

1. Work done is to product of force with displacement

   i.e., $W = \vec{F} \cdot \vec{\Delta r}$

   $W = \vec{F} \cdot (\vec{r_2} - \vec{r_1})$ *where* $\vec{\Delta r}$ = (*Final value – Initial value*) *is the displacement.*

2. *Power (P)* = $\vec{F} \cdot \vec{v}$

Example 26 :

Find the work done $\vec{F} = 3\hat{i} + 2\hat{j} - \hat{k}$ in displacing particle from A (1, –1, 2) to B (5, 3, 1)

Solution :

$\vec{r}_1 = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{r}_2 = 5\hat{i} + 3\hat{j} + \hat{k}$

$\vec{r_2} - \vec{r_1} = \vec{S} = 4\hat{i} + 4\hat{j} - \hat{k}$

$W = \vec{F} \cdot \vec{S} = 12 + 8 + 1 = 21$ J

CHECK YOUR GRASP

☞ *Is dot product commutative?*

☞ What is the geometrical signification of dot product?

EVALUATE YOURSELF - 4

1. Component of $3\hat{i} + 4\hat{j}$ perpendicular to $\hat{i} + \hat{j}$ and in the same plane as that of $3\hat{i} + 4\hat{j}$ is

   (1) $\frac{1}{2}(\hat{j} - \hat{i})$ (2) $\frac{3}{2}(\hat{j} - \hat{i})$ (3) $\frac{5}{2}(\hat{j} - \hat{i})$ (4) $\frac{7}{2}(\hat{j} - \hat{i})$

2. A particle gets displaced with a velocity $(\hat{i} + 2\hat{j} + 5\hat{k})$ m/s due to force $(2\hat{i} - 3\hat{j} + 16\hat{k})$N in 4 second. The power in watts is

   (1) 72 (2) 24 (3) 216 (4) 76

3. If $\vec{A} = \vec{B} - \vec{C}$, then the angle between $\vec{A}$ and $\vec{B}$ is

(1) $\cos^{-1}[(A^2 + B^2 - C^2)/2AB)]$ (2) $\sin^{-1}[(A^2 + B^2 - C^2)/2AB)]$

(3) $\tan^{-1}[(A^2 + B^2 - C^2)/2AB)]$ (4) None of these

4. The component of a vector is

(1) Always less than its magnitude

(2) Always greater than its magnitude

(3) Always equal to its magnitude

(4) None of these

5. A particle of mass 0.5 units moves from a point (3, –4, 5) to the point (–2, 6, –4) while a force of 2i + 3j – k acts on it, calculate the work done

(1) 45 units (2) 60 units (3) 29 units (4) 80 units

6.. A force $\left(3\hat{i}+2\hat{j}\right)N$ displaces on object through $\left(2\vec{i}-3\hat{j}\right)m$. The work done by the force is

(1) zero (2) 5 J (3) 12 J (4) 13 J

7. If two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $-4\hat{i}-6\hat{j}-\lambda\hat{k}$ are anti parallel to each other then value of l be

(a) 0 (b) –2 (c) 3 (d) 4

8. A vector $\vec{P}_1$ is along the positive x-axis. If its vector product with another vector $\vec{P}_2$ is zero then $\vec{P}_2$ could possibly be

(1) $4\hat{j}$ (2) $-4\hat{i}$ (3) $\left(\hat{j}+\hat{k}\right)$ (4) $-\left(\hat{i}+\hat{j}\right)$

9. Consider two vectors $\vec{F}_1 = 2\hat{i}+5\hat{k}$ and $\vec{F}_2 = 3\hat{i}+4\hat{k}$. The magnitude of the scalar product of these vectors is

(1) 20 (2) 23 (3) $5\sqrt{33}$ (4) 26

ANSWERS

1. (1)	2. (4)	3. (1)	4. (4)	5. (3)
6. (1)	7. (2)	8. (2)	9. (4)	

CROSS PRODUCT OR VECTOR PRODUCT

Mathematically,

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

Direction of $\hat{n}$ is given by RHTR (stated earlier)

$\hat{n}$ indicates direction of $\vec{A}\times\vec{B}$ and $-\hat{n}$ indicates direction of $\vec{B}\times\vec{A}$.

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

So, cross product is Non-commutative in nature.

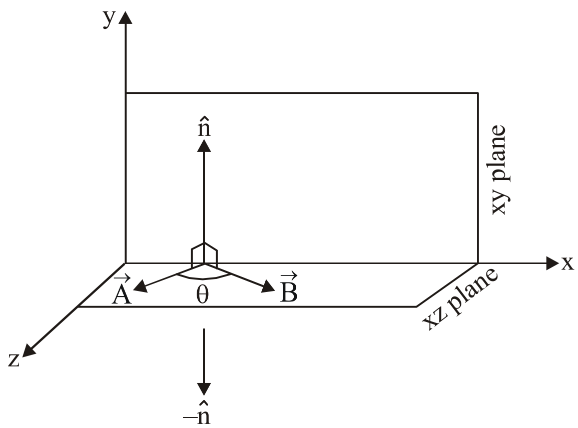
$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n} \Rightarrow |\vec{A}\times\vec{B}| = AB\sin\theta\,\hat{n}$

$\Rightarrow |\vec{B}\times\vec{A}| = BA\sin\theta$

$\therefore |\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$

Also, $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta} = \dfrac{\hat{A}\times\hat{B}}{\sin\theta}$

where, $\hat{n}$ indicates direction of $\vec{A}\times\vec{B}$.



CONCEPT BOOSTER

(a) $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$

(b) $\hat{n}$ *is a new vector perpendicular to* $\vec{A}$ *as well as* $\vec{B}$

$\hat{n}\cdot\vec{A} = 0$

$\hat{n}\cdot\vec{B} = 0$

{Perpendicular vector have dot product = 0}

Hence, $(\vec{A}\times\vec{B})$ is a new vector perpendicular to $\vec{A}$ as well as $\vec{B}$

(c) $\hat{n}$ *is perpendicular to* $\vec{A}$ *as well as* $\vec{B}$

(where $\hat{n}$ indicates direction of $(\vec{A}\times\vec{B})$)

$\therefore$ $\vec{A}\times\vec{B}$ is a new vector $\perp$ to $\vec{A}$ and $\vec{B}$

$\Rightarrow (\vec{A}\times\vec{B})\cdot\vec{A} = 0$ or $(\vec{B}\times\vec{A})\cdot\vec{A} = 0$

$\Rightarrow (\vec{A}\times\vec{B})\cdot\vec{B} = 0$ or $(\vec{B}\times\vec{A})\cdot\vec{B} = 0$

$\Rightarrow (\vec{A}\times\vec{B})\cdot(\vec{A}\pm\vec{B}) = 0$ or $(\vec{B}\times\vec{A})\cdot(\vec{A}\pm\vec{B}) = 0$

PROPERTIES OF VECTOR PRODUCT

(a) Vector product is non commutative i.e.,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \Rightarrow \quad (\vec{A} \times \vec{B}) + (\vec{B} \times \vec{A}) = \vec{0}$$

(b) Cross product is distributive with respect to sum i.e.,

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

CAUTION

$$\vec{A} \times \vec{B} + \vec{C} \times \vec{A} \neq \vec{A} \times (\vec{B} + \vec{C})$$

Instead,

$$\vec{A} \times \vec{B} + \vec{C} \times \vec{A} = \vec{A} \times \vec{B} - \vec{A} \times \vec{C}$$

$$\Rightarrow \quad \vec{A} \times \vec{B} + \vec{C} \times \vec{A} = \vec{A} \times (\vec{B} - \vec{C})$$

(c) If $\vec{A}$ and $\vec{A}$ are parallel, i.e., $\theta = 0°$, then

$$\vec{A} \times \vec{B} = AB \sin 0 \, \hat{n} \quad \Rightarrow \quad \vec{A} \times \vec{B} = \vec{0}$$

Vectors parallel $\Leftrightarrow$ cross-product equal to $\vec{0}$

(d) If $\vec{A}$ and $\vec{A}$ are antiparallel, i.e., $\theta = 180°$, then

$$\vec{A} \times \vec{B} = AB \sin (180°)\hat{n}$$

$$\Rightarrow \quad \vec{A} \times \vec{B} = \vec{0} \quad \because \quad \sin 180 = 0°$$

Vectors parallel $\Leftrightarrow$ cross-product equal to $\vec{0}$

(e) $\vec{A} \times \vec{A} = AA \sin 0 \, \hat{n}$

$$\vec{A} \times \vec{A} = \vec{0}$$

i.e., cross product of vector with itself is $\vec{0}$

(f) $\hat{i} \times \hat{j} = \vec{0}, \hat{j} \times \hat{j} = \vec{0}, \hat{k} \times \hat{k} = \vec{0}$

(g) $\hat{i} \times \hat{j} = \hat{k}, \hat{k} \times \hat{i} = \hat{j}, \ \hat{j} \times \hat{k} = \hat{i}$

For a right handed triade system.

Curl fingers from x to y, thumb gives direction of z.

Curl fingers from y to z, thumb gives direction of x.

Curl fingers z to x, thumb gives direction of y.

(h) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\Rightarrow \vec{A}\times\vec{B} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\Rightarrow \vec{A}\times\vec{B} = \hat{i}(A_yB_z - B_yA_z) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - B_xA_y)$$

(i) Let $\vec{A} = A_x\hat{i} + A_t\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

If $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = k\,(k>0)$, then $\vec{A}$ is parallel to $\vec{B}$

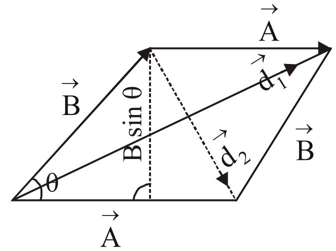
Else if $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = k\,(k<0)$, then $\vec{A}$ is antiparallel to $\vec{B}$

GEOMETRICAL INTERPRETATION OF CROSS PRODUCT

Half of magnitude of cross product equals to area of triangle with adjacent sides $\vec{A}$ and $\vec{B}$.

Area $= \frac{1}{2}$ *(A) (B sin θ)* $= \frac{1}{2}$ *(AB sin θ)*

$\Rightarrow$ *Area of triangle* $= \frac{1}{2}|\vec{A}\times\vec{B}|$



Area of parallelogram = (Base) × (Perpendicular distance between parallel sides)

$= A\,(B \sin\theta) = AB \sin\theta$

Also, area of parallelogram $= |\vec{A} \times \vec{B}| = \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

CONCEPT BOOSTER

So, half of the modulus of cross product equals the area of the triangle with adjacent sides $\vec{A}$ and $\vec{B}$ and magnitude of cross products equals the area of the parallelogram with adjacent sides $\vec{A}$ and $\vec{B}$.

PHYSICAL INTERPRETATION

(a) $\vec{\tau} = \vec{r} \times \vec{F} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}$

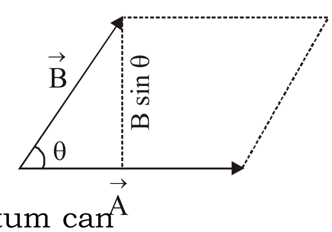
τ (read as tau) i.e., Torque in (Physics)

$\vec{r}$ is the distance of point of application of force from axis of rotation (A.O.R.) and F is force.

t is called moment of force.

(b) $\vec{L} = \vec{r} \times \vec{p}$

$\vec{L}$ = Angular momentum also called moment of linear momentum.



$\vec{p}$ = Linear momentum = $m\vec{v}$

$\vec{p} = m\vec{v}$

Since, mass is a scalar therefore momentum can be reads as mass times velocity.

$\therefore \quad \vec{L} = \vec{r} \times (m\vec{v}) = m\,(\vec{r} \times \vec{v})$

Example 27:

If $\bar{r} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{\omega} = \hat{i} - \hat{j} + \hat{k}$ find $\vec{v}$

Solution :

$\vec{v} = \vec{\omega} \times \vec{r}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$(2+1)\hat{i}-(1-1)\hat{j}+(-1-2)\hat{k}=3\hat{i}-3\hat{k}$$

Example 28 :

Find value of a and b for which $\vec{a}=\alpha\hat{i}-3\hat{j}+2\hat{k}$ parallel to $\vec{b}=2\hat{i}-\beta\hat{j}+\hat{k}$

Solution :

$$\frac{\alpha}{2}=\frac{-3}{\beta}=\frac{2}{1}$$

$\alpha=4 \quad \beta=-3/2$

Example 29:

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them

$(\vec{a}\,.\,\vec{b})^2-(\vec{a}\times\vec{b})^2=$

(1) 1 (2) 0 (3) $\cos 2\theta$ (4) $a^2b^2\cos 2\theta$

Solution :

$a^2b^2\cos^2\theta-a^2b^2\sin^2\theta=a^2b^2(\cos^2\theta-\sin^2\theta)$

$=a^2b^2\cos^2 2\theta$ (d) is correct.

Example 30 :

Find the moment of the force $\vec{F}=2\hat{i}+\hat{j}+3\hat{k}$ acting at (4, 3, –2) about (1, –1, 2).

Solution :

$\vec{\tau}=\vec{r}\times\vec{F}$

$\vec{r}=\vec{r}_a-\vec{r}_b=3\hat{i}+4\hat{j}-4\hat{k}$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -4 \\ 2 & 1 & 3 \end{vmatrix}=16\hat{i}-17\hat{j}-5\hat{k}$$

CHECK YOUR GRASP

☞ *Is cross product distributive?*

☞ *What is the geometrical signification of cross product?*

EVALUATE YOURSELF - 5

1. Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at angle θ. Which of the following is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$?

(1) $\frac{\vec{A}\times\vec{B}}{\vec{A}\cdot\vec{B}}$ (2) $\frac{\hat{A}\times\hat{B}}{\sin\theta}$ (3) $\frac{\vec{A}\times\hat{B}}{AB\sin\theta}$ (4) $\frac{\hat{A}\times\vec{B}}{AB\sin\theta}$

2. What is the torque of the force, when $\vec{F}=2\hat{i}-3\hat{j}+4\hat{k}$ newton acts at the point $\vec{r}=3\hat{i}+2\hat{j}+3\hat{k}$ metre about the origin?

(1) $6\hat{i}-6\hat{j}+12\hat{k}$ (2) $17\hat{i}-6\hat{j}-13\hat{k}$ (3) $-6\hat{i}+6\hat{j}-12\hat{k}$ (4) $-17\hat{i}+6\hat{j}+12\hat{k}$

3. If $|\vec{A}\times\vec{B}|=0$, which of the following is not possible?

(1) Either $\vec{A}$ or $\vec{B}$ is zero (2) Angle between $\vec{A}$ and $\vec{B}$ is zero

(3) Angle between $\vec{A}$ and $\vec{B}$ is 90° (4) Angle between $\vec{A}$ and $\vec{B}$ is 180°

4. The adjacent sides of a parallelogram are represnted by co-initial vector $2\hat{i}+3\hat{j}$ and $\hat{i}+4\hat{j}$. The area of the parallelogram is

(1) 5 units along *z*-axis (2) 5 units in *x*-*y* plane

(3) 3 units in *x*-*z* plane (4) 3 unit in *y*-*z* plane

5. What is the angle between $\hat{P}+\hat{Q}$ and $\hat{P}\times\hat{Q}$?

(1) 0 (2) $\pi/4$ (3) $\pi/2$ (4) π

6. Which of the following is the unit vector perpendicular to $\vec{A}$ and $\vec{B}$

(A) $\frac{\hat{A}\times\hat{B}}{AB\sin\theta}$ (B) $\frac{\hat{A}\times\hat{B}}{AB\cos\theta}$ (C) $\frac{\vec{A}\times\vec{B}}{AB\sin\theta}$ (D) $\frac{\vec{A}\times\vec{B}}{AB\cos\theta}$

7. Let $\vec{A}=\hat{i}A\cos\theta+\hat{j}A\sin\theta$ be any vector. Another vector $\vec{B}$ which is normal to *A* is

(A) $\hat{i}B\cos\theta+\hat{j}B\sin\theta$ (B) $\hat{i}B\sin\theta+\hat{j}B\cos\theta$

(C) $\hat{i}B\sin\theta-\hat{j}B\cos\theta$ (D) $\hat{i}B\cos\theta-jB\sin\theta$

8 The angle between two vectors given by $6\hat{i}+6\hat{j}-3\hat{k}$ and $7\hat{i}+4\hat{j}+4\hat{k}$ is

(A) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\cos^{-1}\left(\frac{5}{\sqrt{3}}\right)$ (C) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (D) $\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)$

9. A vector $\vec{A}$ points vertically upward and $\vec{B}$ points towards north. The vector product $\vec{A}\times\vec{B}$ is

(A) Zero (B) Along west (C) Along east (D) Vertically downward

10. Angle between the vectors $(\hat{i}+\hat{j})$ and $(\hat{j}-\hat{k})$ is

(A) 90° (B) 0° (C) 180° (D) 60°

ANSWERS

1. (2)	2. (2)	3. (3)	4. (1)	5. (3)
6. (3)	7. (3)	8. (4)	9. (2)	10. (4)

EXERCISE – 1

1. What is the value of $\frac{dy}{dx}$ at x = 2, if $y = 3x^2 - 2x + \pi$
   (1) 5 (2) 10
   (3) 12 (4) π
2. Find the value of integration $\int_0^2 (3x^2 + 2x)dx$
   (1) 8 (2) 10
   (3) 12 (4) 16
3. Find the integral of $I = \int x e^{x^2} dx$
   (1) $e^{x^2} + c$ (2) $\frac{e^{x^2}}{2} + c$
   (3) $xe^{x^2} + c$ (4) $xe^x + c$
4. If the sum of two unit vectors is a unit vector, then the magnitude of their difference is
   (1) 1 (2) $\sqrt{3}$
   (3) $\frac{1}{\sqrt{3}}$ (4) 2
5. 91 $|\vec{a}| = 3$ & $|\vec{b}| = 2$, then $\vec{a}.\vec{b}$ can't be
   (1) 3 (2) 2
   (3) 6 (4) 12
6. $|\vec{a}| = 3$ & $|\vec{b}| = 2$, then $|\vec{a} \times \vec{b}|$ can't be
   (1) 12 (2) 2
   (3) 3 (4) 6
7. Let $\vec{A} = \hat{i}A\cos\theta + \hat{j}A\sin\theta$, be any vector. Another vector $\vec{B}$ which is normal to $\vec{A}$ is:
   (1) $\hat{i}B\cos\theta + \hat{j}B\sin\theta$
   (2) $\hat{i}B\sin\theta + \hat{j}B\cos\theta$
   (3) $\hat{i}B\sin\theta - \hat{j}B\cos\theta$
   (4) $\hat{i}A\cos\theta - \hat{j}A\sin\theta$
8. If the vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other, then the positive value of a is
   (1) 3 (2) 2
   (3) 1 (4) 0
9. The area of parallelogram formed by the vectors $\vec{a} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{b} = 5\hat{i} - 6\hat{j} + 6\hat{k}$
   (1) 7 (2) $\sqrt{104}$
   (3) $\sqrt{219}$ (4) $\sqrt{497}$
10. If $\vec{A}$ is parallel to $\vec{B}$, then
    (1) $\vec{A}.\vec{B} = 0$
    (2) $\vec{A} \times (\vec{A} + \vec{B}) = AB$
    (3) $\vec{B} \times (\vec{A} + \vec{B}) = A^2 + AB$
    (4) $\vec{A} \times \vec{B} = \vec{0}$
11. If $\vec{A}$ and $\vec{B}$ denote the sides of a parallelogram and its area is ½ AB (A and B are the magnitude of $\vec{A}$ and $\vec{B}$ respectively), the angle between $\vec{A}$ and $\vec{B}$ is
    (1) 30° (2) 60°
    (3) 45° (4) 120°
12. A vector $\vec{A}$ points vertically upwards and $\vec{B}$ points towards North. The vector product $\vec{A} \times \vec{B}$ is
    (1) Zero
    (2) Along east
    (3) Along west
    (4) Vertically downwards
13. If two vectors, $\vec{a}$ and $\vec{b}$ are in the same direction, then choose the correct alternative(s)

(1) $\vec{a} = \vec{b}$ (2) $|\vec{a}| = |\vec{b}|$

(3) $\vec{a}.\vec{b} = 0$ (4) $\vec{a} \times \vec{b} = 0$

14. A vector $\vec{a}$ will not change if it is
    (1) multiplied by a number
    (2) Rotated by an angle q
    (3) Shifted parallely to the line of vector
    (4) It is not possible to produce change in $\vec{a}$ by any mathematical operation

15. If the vector sum of 'n' vectors is zero, where n > 2, then choose the correct options
    (1) All the vectors must be collinear
    (2) They can be shifted to form a polygon of n-sides
    (3) All the vectors must be in same direction
    (4) The magnitude of all the vectors must be zero

16. Which of the following vector(s) do not have a definite direction
    (1) Unit vector (2) Zero vector
    (3) Position vector (4) Gravity

17. Which of the following can not be possible angle(s) between two vectors
    (1) 110° (2) $\frac{3\pi}{2}$ radian
    (3) $\frac{\pi}{2}$ radian (4) 0°

18. If $\hat{a}$ is a unit vector along the force $\vec{F} = (10N)\hat{i}$, then the wrong option is
    (1) $|\hat{a}| = 1$
    (2) S.I unit of $\hat{a}$ is N
    (3) $\hat{a}$ is dimensionless
    (4) $\hat{a} = \hat{i}$

19. Which vector is used only to represent a specific direction in space
    (1) Zero vector
    (2) Unit vector
    (3) Position vector
    (4) Displacement vector

20. What is the magnitude of the vector $\vec{a} = (4\hat{i} + 2\hat{j} - 4\hat{k})m$
    (1) 10m (2) 2m
    (3) 6m (4) 36m

21. If $\vec{a} = (6\hat{i})m$ and $\vec{b} = (8\hat{j})m$, then choose the incorrect option(s)
    (1) $\vec{a}.\vec{b} = 48$
    (2) $\vec{a} \times \vec{b} = 48\hat{k}$
    (3) $\vec{a} + \vec{b} = (6\hat{i} + 8\hat{j})m$
    (4) $|\vec{a} + \vec{b}| = 10m$

22. The unit vector in the direction of $\vec{a} = (4\hat{i} + 2\hat{j} - 4\hat{k})m$ will be
    (1) $\frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})m$
    (2) $\frac{1}{3}(4\hat{i} + 2\hat{j} - 4\hat{k})m$
    (3) $(\hat{i} + \hat{j} - \hat{k})$
    (4) $\frac{1}{3}(2\hat{i} + \hat{j} - 2\hat{k})$

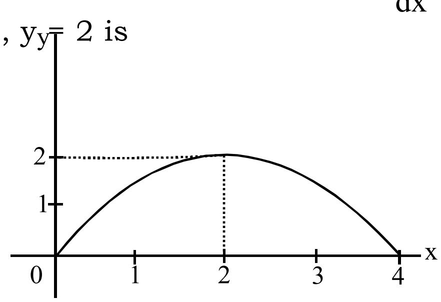
EXERCISE – 2

1. For a particle moving along a straight line, the displacement x depends on time t as $x = \alpha t^3 + \beta t^2 + \gamma t + \delta$. The ratio of its initial acceleration to its initial velocity depends
   (1) Only on α and β
   (2) Only on β and γ
   (3) Only on α and γ
   (4) Only on α

2. A point moves along a straigth line with retardation 'f' which depends on the velocity as $f = a\sqrt{V}$. Where 'a' is a constant. If the initial velocity of point is V_o. What distance does it traverse before it stops?
   (1) $\frac{2V_0^2}{a}$ (2) $\frac{2\sqrt{V_0}}{a}$
   (3) $\frac{2V_0^{3/2}}{3a}$ (4) $\frac{V_0^2}{4a}$

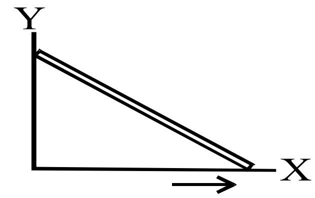
3. The distance (x) covered by a particle moving on a straight line path at any instant is given by $t = ax^2+bx$. If the instantaneous velocity of the particle be v. What is the acceleration of the particle?
   (1) 2 av^3 (2) –2 av^3
   (3) 2 av^2 (4) –2 av^2

4. The graph of variable y vs variable x is as shown . The value of $\frac{dy}{dx}$ at x = 2, y = 2 is



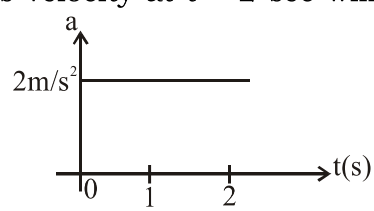
   (1) 1 (2) Zero
   (3) ∞ (4) None of these

5. The rod shown in fig starts slipping. Find the speed of lower end if the speed of upper end is $\sqrt{2}$ m/s when it makes an angle 45° with the x axis



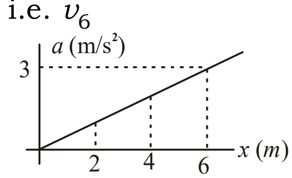
   (1) 1 m/s (2) $\sqrt{2}$ m/s
   (3) $-\sqrt{2}$ m/s (4) –1 m/s

6. The acceleration time graph of a particles is as shown in figure. If initial velocity of the particle is 2m/s its velocity at t = 2 sec will be



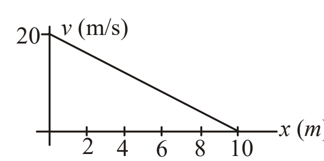
   (1) 6 m/s (2) 4 m/s
   (3) 8 m/s (4) 2 m/s

7. In the graph shown, if velocity at x = 2 m i.e. v_2 = 3 m/s, find velocity at x = 6 m i.e. v_6



   (1) 4 m/s (2) 5 m/s
   (3) 6 m/s (4) 8 m/s

8. In the graph drawn, find acceleration at x = 2 m



   (1) 32 m/s^2 (2) –32 m/s^2
   (3) 8 m/s^2 (4) –8 m/s^2

9. If $v = \sqrt{x^2 + 2x}$ is equation for velocity (v) of a particle as a function of its position (x). Find accelerationat x = 1 m
(1) 1 m/s^2 (2) 2 m/s^2
(3) 4 m/s^2 (4) 8 m/s^2

10. The vector of length l is turned through the angle θ about its tail. What is the change in the position vector of its head?
(1) $l \cos(\theta/2)$ (2) $2l \sin(\theta/2)$
(3) $2l \cos(\theta/2)$ (4) $l \sin(\theta/2)$

11. If $\vec{a}.\vec{b} = 0$ and $|\vec{a} \times \vec{b}| = 1$ then
(1) $\vec{a}$ & $\vec{b}$ must be perpendicular
(2) $\vec{a}$ and $\vec{b}$ must be perpendicular unit vectors
(3) $\vec{a}$ and $\vec{b}$ must be parallel
(4) $\vec{a}$ and $\vec{b}$ must be parallel unit vectors

12. If $\vec{a} + \vec{b} = \vec{c}$ and a = 2, b = 3 and c = 4, then $|\vec{c} - \vec{b} + \vec{a}|$ is equal to
(1) 4 (2) 6
(3) 8 (4) None of these

13. Which of the following is false?
(1) $|\hat{a} + \hat{b}| \leq 2$
(2) $|\hat{a} + \hat{b}| \geq 2$ (3) $|\hat{a} - \hat{b}| \leq 2$
(4) all the statements in 1, 2 & 3 option

14. Which of the following sets of displacements might be capable of returning a car to its starting point
(1) 4, 6, 8 and 15
(2) 10, 30, 50 and 120 km
(3) 5, 10, 30 and 50
(4) 40, 50, 75 and 200 km

15. If $\vec{A} = 4\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - 2\hat{j} - 3\hat{k}$, the angle which $(\vec{A} + \vec{B})$ makes with x-axis is
(1) $\text{Cos}^{-1}\left(\frac{3}{\sqrt{50}}\right)$
(2) $\text{Cos}^{-1}\left(\frac{5}{\sqrt{50}}\right)$
(3) $\text{Cos}^{-1}\left(\frac{4}{\sqrt{50}}\right)$
(4) $\text{Cos}^{-1}\left(\frac{12}{\sqrt{50}}\right)$

16. Vector A = $\hat{i}$ is rotated in anticlock wise direction by an angle 45°, the vector A becomes equal to
(1) $\hat{i} + \hat{j}$ (2) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$
(3) $\hat{i} - \hat{j}$ (4) Remains same

17. If $\vec{a} = \alpha\hat{i} - 2\hat{j} + \hat{k}$ is parallel to $\vec{b} = 3\hat{i} + \beta\hat{j} - \hat{k}$ the α and β are respectively
(1) 3, 2 (2) 2, 3
(3) –3, 2 (4) –2, 3

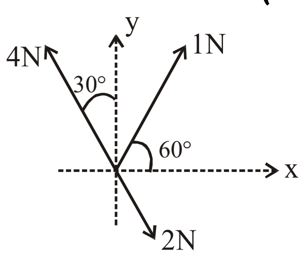
18. Following forces start acting on a particle at rest at the origin of the coordinate system simultaneously
$\vec{F}_1 = -4\hat{i} - 5\hat{j} + 5\hat{k}$ $\vec{F}_2 = 5\hat{i} + 8\hat{j} + 6\hat{k}$
$\vec{F}_3 = -3\hat{i} + 4\hat{j} - 7\hat{k}$ $\vec{F}_4 = 2\hat{i} - 3\hat{j} - 2\hat{k}$
Then the particle will move
(1) In X – Y plane
(2) In Y – Z
(3) In X – Z plane
(4) Along X axis

EXERCISE – 3

1. $\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between them, if $|\vec{A}\times\vec{B}|=\sqrt{3}\left(\vec{A}.\vec{B}\right)$ the value of θ is: **[CBSE 2007]**

(1) 90° (2) 60°
(3) 45° (4) 30°

2. Three forces acting on a body are shown in the figure. To have the resultant force only along the y-direction, the magnitude of the minimum additional force needed is **(CBSE 2008)**

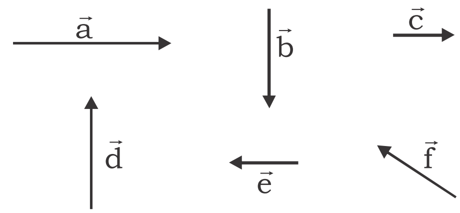


(1) $\sqrt{3}$ N (2) 0.5 N
(3) 1.5 N (4) $\frac{\sqrt{3}}{4}$ N1

3. A particle of mass m is projected with velocity v making an angle of 45° with the horizontal. When the particle land on the level ground the magnitude of the change in its momentum will be **(CBSE 2008)**

(1) Zero (2) 2 mv
(3) mv / $\sqrt{2}$ (4) mv $\sqrt{2}$

4. Six vectors, $\vec{a}$ to $\vec{f}$ have the magnitudes and directions indicated in the figure. Which of the following statements is true? **(CBSE 2010)**



(1) $\vec{d}+\vec{e}=\vec{f}$ (2) $\vec{b}+\vec{e}=\vec{f}$
(3) $\vec{b}+\vec{c}=\vec{f}$ (4) $\vec{d}+\vec{c}=\vec{f}$

5. If vectors $\vec{A}=\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j}$ and $\vec{B}=\cos\frac{\omega t}{2}\hat{i}+\sin\frac{\omega t}{2}\hat{j}$ are functions of time, then the value of t at which they are orthogonal to each other is: **(AIMPT 2015)**

(1) t = 0 (2) $t=\frac{\pi}{4\omega}$
(3) $t=\frac{\pi}{2\omega}$ (4) $t=\frac{\pi}{\omega}$

EXERCISE – 4

Questions with Assertion and Reason

(1) If both **Assertion** and the **Reason** are true and the Reason is a correct explanation of the Assertion.

(2) If both the **Assertion** and the **Reason** are true but the **Reason** is not a correct explanation of the **Assertion**.

(3) If the **Assertion** is true but the **Reason** is false.

(4) If both the **Assertion** and the **Reason** are false.

1. **Assertion**: The perpendicular vector of $(\hat{i}+\hat{j}+\hat{k})$ is $(\hat{i}-2\hat{j}+\hat{k})$.

Reason: Two vectors are perpendicular if their dot product is equal to zero.

2. **Assertion**: When $|\vec{P}+\vec{Q}|=|\vec{P}-\vec{Q}|$, then $\vec{P}$ must be perpendicular to $\vec{Q}$.

Reason: The relation holds only when $\vec{Q}$ is null vector.

3. **Assertion**: If the sum of the two unit vectors is also a unit vector, the magnitude of their diffrence is root of three.

Reason: To find resultant of two vectors, we use square law.

ANSWERS (MATHEMATICAL TOOLS & VECTOR)

EXERCISE – 1

1. (2)	6. (1)	11. (1)	16. (2)	21. (1)
2. (3)	7. (1)	12. (3)	17. (2)	22. (4)
3. (2)	8. (1)	13. (4)	18. (2)	
4. (4)	9. (4)	14. (3)	19. (2)	
5. (4)	10. (4)	15. (2)	20. (4)	

EXERCISE – 2

1. (2)	5. (2)	9. (2)	13. (2)	17. (3)
2. (3)	6. (1)	10. (2)	14. (1)	18. (2)
3. (2)	7. (2)	11. (1)	15. (2)	
4. (2)	8. (2)	12. (1)	16. (2)	

EXERCISE – 3

1. (2)	2. (2)	3. (4)	4. (1)	5. (4)

EXERCISE – 4

1. (1)	2. (3)	3. (3)

HINTS AND SOLUTIONS

EXERCISE – 1

1. Q $\frac{dy}{dx} = 6x - 2$

   $\therefore$ at $x = 2;\ \frac{dy}{dx} = 10$

2. Q $\int_0^2 \left(3x^2 + 2x\right)dx = \left[x^3 + x^2\right]_0^2 = 12$

3. Let $e^{x^2} = t$

   $\therefore\ 2x\,e^{x^2}dx = dt$

   $\therefore\ xe^{x^2} = \frac{dt}{2}$

   $\therefore\ I = \int \frac{dt}{2} = \frac{t}{2} + c = \frac{e^{x^2}}{2} + c$

20. $|\vec{a}| = \sqrt{16 + 4 + 16}\,m = 6m$

22. $\because\ \hat{a} = \frac{\vec{a}}{|\vec{a}|}$